An Introduction to Copulas
with Application to Structured Credit

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Outline

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1. **Motivation**
2. **Theoretical Framework**
   - Definitions & Key Theorems
   - Other results
   - Examples
   - Copulas and Latent Variable Modeling

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3. **Financial application : an example**
   - Credit Default Swap (CDS)
   - Collaterallized Debt Obligation (CDO)
   - Valuing CDOs using copulas

About this presentation

First of all

- I am not an expert in this field...
- But, I've learned key copula concepts, used copula modeling (a bit), and know Machine Learning (a bit)
- My 1st goal : to convince you that copula may also be used in Machine Learning
- My 2nd goal : to show you how copulas can be used in Finance

Estimating joint CDF

Let $\mathbf{X} = [X_1, \ldots, X_N] \in \mathbb{R}^N$ and $\mathbf{F_X}$ the CDF of $\mathbf{X}$

Q Assume $\mathbf{x}(1), \ldots, \mathbf{x}(T)$, $\mathbf{x}(i) \in \mathbb{R}^N$. How to compute $\hat{\mathbf{F}}_\mathbf{X} \approx \mathbf{F_X}$?

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A Easy: $\mathbf{F_X}(\cdot) \sim \mathbf{F}(\cdot; \mathbf{\Theta})$ and $\mathbf{\Theta}^\star$ given by ML, MAP,...

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- **Q** How many tractable $N-$variate parametric families $\mathbf{F}(\cdot; \boldsymbol{\Theta})$ do you know ?

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- **A** Easy: $\mathbf{F_X}(\cdot) \sim \mathbf{F}(\cdot; \mathbf{\Theta})$ and $\mathbf{\Theta}^\star$ given by ML, MAP,...
- **Q** How many tractable $N-$variate parametric families $\mathbf{F}(\cdot; \mathbf{\Theta})$ do you know ?
- **A** Estimating margins F_i will be easier : T samples for 1 dimension $\Rightarrow$ estimates of F_i more reliable than $\hat{\mathbf{F}}$.

Estimating joint CDF

Three choices for the computation of $\hat{\mathbf{F}}, \hat{F}_1, \ldots, \hat{F}_N$:

- Estimate $\hat{\mathbf{F}}$ and infer $\hat{F}_i$ by $\int_{X_j, j \neq i} \hat{\mathbf{F}}(\mathbf{X}) \prod_{j \neq i} dX_j$

  $\Rightarrow$ but recall that $\hat{F}_i \approx F_i$ more reliable than $\hat{\mathbf{F}} \approx \mathbf{F}$

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- Estimate $\hat{\mathbf{F}}$ and $\hat{F}_i$ separately
  $\Rightarrow$ but then inconsistencies : $\int_{X_j, j\neq i} \hat{\mathbf{F}}(\mathbf{X}) \prod_{j\neq i} dX_j \neq \hat{F}_i(X_i)$

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- Estimate F_i, find a constrained optimization technique to estimate $\hat{\mathbf{F}} | \{\hat{F}_i\}$
  $\Rightarrow$ but how to achieve that ? copula may help

Measuring dependence

- How would you measure the dependency bw X_i *due to the coupling only* (i.e. independent of the margins) ?

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- Most of measures are *margin-sensitive*
- insensitive to margins ? $\Rightarrow$ copula may help

Latent variable model

- Let $X_1, \ldots, X_N$ be dependent RVs, $X_i \sim F_i$
- Assume each has an idiosynchratic & systemic part

$$\begin{aligned} \hat{X}_i &\doteq \alpha_i \tilde{X}_i + \mathbf{\Gamma}_i \mathbf{Z}^T \quad , \qquad \Rightarrow X_i \doteq F_i^{-1}(F_{\hat{X}_i}(\hat{X}_i)) \\ \hat{X}_j &\doteq \alpha_j \tilde{X}_j + \mathbf{\Gamma}_j \mathbf{Z}^T \quad , \qquad \Rightarrow X_j \doteq F_j^{-1}(F_{\hat{X}_j}(\hat{X}_j)), \end{aligned}$$

  with $\mathbf{\Gamma}_{i,j} \in \mathbb{R}^K$, Z_i i.i.d., $Z_i \perp \tilde{X}_j$, $\tilde{X}_i \perp \tilde{X}_{j \neq i}$,
  $\mathrm{Cov}(\hat{X}_i, \hat{X}_j) = \mathbf{\Gamma}_i \mathbf{\Gamma}_j^T$
- Choosing the latent variables $\Leftrightarrow$ choosing the copula associated to $\mathbf{F_X}$

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Outline

Copulas : Definition

Define the N-dim unit-cube $I^N \doteq \underbrace{[0,1] \times \ldots \times [0,1]}_{N \ times}$

A Copula is a function $C : I^N \rightarrow I$ satisfying the following:

- C is grounded (that is $C(\mathbf{u}) = 0$ if $\exists i \in \{1, \ldots, N\} : u_i = 0$);
- each of the one dimensional margins C_i of C are the identity function $C_i(u_i) = u_i$
- C is N-increasing

Copula as a CDF

Let **U** be a N-dimensional uniform random variable on I_N

- Domain and Image :
  $F_{\mathbf{U}}(\mathbf{x}) : \Omega(\mathbf{U}) = I_N = Dom(C) \rightarrow I = Im(C)$;
- Groundness : $F_{U_1,U_2}[0,x] = 0$:
  $\Pr[U_1 \leqslant 0, U_2 \leqslant x] = \Pr[U_2 \leqslant x | U_1 \leqslant 0] \Pr[U_1 \leqslant 0] = 0$;
- $\Pr[U_i \leqslant u] = F_{U_i}(u) = u$ (CDF of U_i is the identity function)
- N-increasingness A 1-dim CDF must be increasing, a N-dim CDF (N>1) must be N-increasing.

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Copula definition = necess. and suff. conditions for a function to be the CDF of **U**

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The set of N-dim copulas is noted $C \in \mathcal{C}(N)$

Sklar's Theorem
Probabilistic interpretation of copulas

- For any N-dim CDF $\mathbf{F}$ with margins F_i, $\exists\ C \in \mathcal{C}(N)$ s.t.

$$C(F_1(x_1), \ldots, F_N(x_N)) \overset{not.}{=} C(\{F_i(x_i)\}) = \mathbf{F}(\{x_i\}) = \mathbf{F}(\mathbf{X})$$

$\Rightarrow$ C is *unique* if the margins are continuous
Any CDF is a copula (potentially up to proper univariate mappings involving the margins)

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- For any $C \in \mathcal{C}(N)$, $\exists F_{\mathbf{X}}$ with invertible margins $\{F_i(X_i)\}$ s.t.

$$C(\{u_i\}) = F_{\mathbf{X}}(\{F_i^{-1}(u_i)\})$$

Any copula is a CDF

Fréchet-Hoeffding's Theorem
Upper & lower bounds for copulas

- Let $C \in \mathcal{C}(N)$ and

$$
\begin{aligned}
C^{-}(\mathbf{u}) &\doteq \max\left(\sum_{i=1}^{N} u_i - (N-1), 0\right) \\
C^{+}(\mathbf{u}) &\doteq \min_{i}(u_i)
\end{aligned}
$$

Then,

$$C^{-} \prec C \prec C^{+}$$

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- $C^{-} \in \mathcal{C}(N)$ iff $N = 2$ and $C^{+} \in \mathcal{C}(N)$ $\forall N \geqslant 2$

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The copula density

$$
\begin{aligned}
c(\mathbf{u}) &\doteq \left.\frac{\partial^N C(\mathbf{v})}{\partial v_1, \ldots \partial v_N}\right|_{\mathbf{u}} = \left.\frac{\partial^N F(\{F_i^{-1}(v_i)\})}{\partial v_1, \ldots \partial v_N}\right|_{\mathbf{u}} \\
&= f(\{F_i^{-1}(v_i)\}) \prod_{i=1}^{N} \left.\frac{dF_i^{-1}(v_i)}{dv_i}\right|_{\mathbf{u}} \\
&= f(\{F_i^{-1}(v_i)\}) \prod_{i=1}^{N} \left.\frac{1}{dF_i(x)/dx|_{x=F_i^{-1}(v_i)}}\right|_{\mathbf{u}}
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- The change of variable $u_i \leftarrow F_i(x_i)$ leads to

$$c(\{F_i(x_i)\}) = \frac{f(\mathbf{x})}{\prod_{i=1}^{N} f_i(x_i)} \Leftrightarrow f(\mathbf{X}) = c(\{F_i(X_i)\}) \times f(\mathbf{X}^{\perp})$$

- Consequence : $MI(\mathbf{X}) = E_{\mathbf{X}}[\log\{c(\{F_i(X_i)\})\}]$

The Fréchet distance

- Define $K(N) \doteq \int_{I^N} C^+(\mathbf{u}) - C^-(\mathbf{u}) d\mathbf{u} = \frac{N!-1}{(N+1)!}$
- By Fréchet : $\int_{I^N} |C_1(\mathbf{u}) - C_2(\mathbf{u})| d\mathbf{u} \leqslant K(N)\ \forall C_1, C_2 \in \mathcal{C}(N)$

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- Define the Fréchet divergence bw copulas:

$$D(C_1, C_2) \doteq \frac{\int_{I^N} |C_1(\mathbf{u}) - C_2(\mathbf{u})| d\mathbf{u}}{K(N)}$$

- The Fréchet divergence is a distance satisfying $0 \leqslant D(C_1, C_2) \leqslant 1$
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- $\Big(\mathcal{C}(N), D(\cdot, \cdot)\Big)$ is a metric space
- This is a measure of distance bw coupling functions !

The Sklar-Fréchet (SF) distance

- Assume two N-dim CDFs $\mathbf{F}_1, \mathbf{F}_2$ with the same margins F_i
- Assume $\mathbf{F}_1(\mathbf{x}) \doteq C_1(\{F_i(x_i)\})$ and $\mathbf{F}_2(\mathbf{x}) \doteq C_2(\{F_i(x_i)\})$
- Define $\mathbf{X}^{\perp} \sim \prod_{i=1}^{N} F_i(X_i)$

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- Define $\mathbf{X}^{\perp} \sim \prod_{i=1}^{N} F_i(X_i)$
- Using Sklar's Theorem:

$$D(C_1, C_2) = \frac{\int_{\mathbf{x}\in\mathbf{\Omega}} |C_1(\{F_i(x_i)\}) - C_2(\{F_i(x_i)\})| \prod_{i=1}^{N} dF_i(x_i)}{K(N)}$$

$$= \frac{E_{X^{\perp}}[|\mathbf{F}_1 - \mathbf{F}_2|]}{K(N)}$$

A new dependence measure

- Define the product copula $C^{\perp}(\mathbf{u}) \doteq \prod_{i=1}^{N} u_i$
- This is the coupling corresponding to independence : $C^{\perp}(\{F_i(x_i)\}) = F_{\mathbf{X}^{\perp}}(\mathbf{x}) = \prod_{i=1}^{N} F_i(x_i)$

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- $D(C, C^{\perp})$ is the distance of the *coupling* C to the *independent coupling*
- But : $D(C, C^{\perp}) = \frac{E_{X^{\perp}}[|\mathbf{F} - \prod_{i=1}^{N} F_i|]}{K(N)}$
- $\frac{E_{X^{\perp}}[|\mathbf{F} - \prod_{i=1}^{N} F_i|]}{K(N)}$ is a dependence measure bw the X_i being independent of the margins (as opposed e.g. to MI)

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The Gaussian Copula [Definition]

The N-dim std Gaussian copula with cov matrix $\mathbf{\Sigma}$

$$C_{\mathcal{N}}(\mathbf{u};\mathbf{\Sigma}) = C_{\mathcal{N}}(u_1,\ldots,u_N;\mathbf{\Sigma})$$

is defined as

- The *I love LaTeX* style

$$\int_{x_1=-\infty}^{\Phi^{-1}(u_1)} \cdot \int_{x_N=-\infty}^{\Phi^{-1}(u_N)} \frac{1}{(2\pi)^{N/2}\sqrt{|\mathbf{\Sigma}|}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)} dx_1 \cdot dx_N$$

$\Phi \sim$ std 1-dim Gaussian CDF, and Φ^{-1} its inverse

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  $\Phi \sim$ std 1-dim Gaussian CDF, and Φ^{-1} its inverse

- The *even Word can do it* style

$$\boldsymbol{\Phi}(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_N);\boldsymbol{\Sigma})$$

  $\boldsymbol{\Phi}(\cdot;\boldsymbol{\Sigma}) \sim$ std N-dim Gaussian CDF with cov matrix $\boldsymbol{\Sigma}$

The Gaussian Copula [Interpretation]

- The Probability Integral Transform

$$F_{X_i}(X_i) \sim \mathcal{U}$$

- Let's evaluate the Gaussian copula at

$$\mathbf{u} = [u_1, \ldots, u_N] \text{ with } u_i \leftarrow F_{X_i}(x_i)$$

- This leads to

$$C_{\mathcal{N}}(\mathbf{u}; \mathbf{\Sigma}) = \mathbf{\Phi}(\Phi^{-1}(\{F_{X_i}(x_i)\})); \mathbf{\Sigma}) = \mathbf{\Phi}(\{\Phi^{-1}(u_i)\}; \mathbf{\Sigma})$$

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- Now assume $X_i \sim \mathcal{N}(0,1)$ ($\Leftrightarrow F_{X_i}(x_i) = \Phi(x_i)$):

$$\begin{aligned} C_{\mathcal{N}}(\mathbf{u}; \mathbf{\Sigma}) &= \mathbf{\Phi}(\{\Phi^{-1}(\Phi(x_i))\}); \mathbf{\Sigma}) \\ &= \mathbf{\Phi}(\{x_i\}); \mathbf{\Sigma}) \end{aligned}$$

- $\Rightarrow C_{\mathcal{N}}(\{\Phi(x_i)\}; \mathbf{\Sigma}) = \mathbf{\Phi}(\{x_i\}; \mathbf{\Sigma})$

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Other families of copulas [Most popular ones]

name	def	Ran(θ)	τ
Product Copula	$C^{\perp}(u_1, u_2) = u_1 u_2$	N.A.	0
FGM	$C^{FGM}(u_1, u_2) = u_1 u_2(1 + \theta(1 - u_1)(1 - u_2))$	[-1,1]	$2\theta/9$
Fréchet	$pC^m + qC^M + (1 - (p + q))C^{\perp}$	N.A.	N.A.
Clayton	$C^{cl}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1\theta}$	$(0, \infty)$	$\frac{\theta}{\theta+2}$
Frank	$\frac{-1}{\theta} \log\left(1 + \frac{1+(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right)$	$(-\infty, \infty)$	$\exists$
Gaussian	...	...	...
Student-t	...	...	...
Double-t	...	...	...
...	...	...	...

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Coupling Margins using independent Latent Variables

Assume $\mathbf{V} \sim \mathbf{G_V}$, $\mathbf{G_V}$ unknown, margins G_i known.

- Assume $V_i \sim G_i$, $i \in \{1, \ldots, N\}$. Suppose $X_i \sim F_i$. Then :
  - $\exists \theta_i s.t. \Pr[V_i \leqslant v_i] = \Pr[X_i \leqslant \theta_i]$.

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  - $\exists \theta_i s.t. \Pr[V_i \leqslant v_i] = \Pr[X_i \leqslant \theta_i]$.
  - $\Pr[\{X_i \leqslant \underbrace{F_i^{-1}(G_i(v_i))}_{\doteq \theta_i}\}] = F_i(\theta_i) = F_i \circ F_i^{-1}(G_i(v_i)) = G_i(v_i)$

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  - $\exists \theta_i s.t. \Pr[V_i \leqslant v_i] = \Pr[X_i \leqslant \theta_i]$.
  - $\Pr[\{X_i \leqslant \underbrace{F_i^{-1}(G_i(v_i))\}}_{\doteq \theta_i}] = F_i(\theta_i) = F_i \circ F_i^{-1}(G_i(v_i)) = G_i(v_i)$
- Consider $\mathbf{X} \sim \mathbf{F}^{\perp} = \prod_{i=1}^{N} F_i$:
  - $\Pr[\{V_i \leqslant v_i\}] = \Pr[\{X_i \leqslant \theta_i\}] = \prod_{i=1}^{N} G_i(v_i)$
- $\mathbf{V}$ is modelled through independent latent variables ($X_i \perp X_j$)
- This model yields the product copula

Coupling Margins using Gaussian Latent Variables

Assume $V_i \sim G_i$ (known) and CDF $\mathbf{G_V}$ to be modelled

- Consider $X_i \sim \Phi$ ($\mathbf{X} \sim \mathbf{F_X}$) and let's Gaussianize :
  - $\Pr[\{V_i \leqslant v_i\}] = \Pr[\{X_i \leqslant \theta_i\}]$, $\theta_i \doteq \Phi^{-1}(F_i(v_i))$
  - dependence bw $V_i \Leftrightarrow$ dependence bw X_i

Coupling Margins using Gaussian Latent Variables

Assume $V_i \sim G_i$ (known) and CDF $\mathbf{G_V}$ to be modelled

- Consider $X_i \sim \Phi$ ($\mathbf{X} \sim \mathbf{F_X}$) and let's Gaussianize :
  - $\Pr[\{V_i \leqslant v_i\}] = \Pr[\{X_i \leqslant \theta_i\}]$, $\theta_i \doteq \Phi^{-1}(F_i(v_i))$
  - dependence bw $V_i \Leftrightarrow$ dependence bw X_i
- Assume the following coupling model ($Z_i \sim \Phi$, $\tilde{X}_i \sim$ iid Φ):

$$X_i = \sqrt{1 - \Gamma_i \Gamma_i^T} \tilde{X}_i + \Gamma_i \mathbf{Z}^T \ \Leftrightarrow \ \tilde{X}_i = \frac{X_i - \Gamma_i \mathbf{Z}^T}{\sqrt{1 - \Gamma_i \Gamma_i^T}}$$

- $\Rightarrow$ Conditionally on $\mathbf{Z}$, $X_i \perp X_j \ \forall i \neq j$
- $\mathrm{Cov}(X_i, X_j) = \Gamma_i \Gamma_j^T \Rightarrow \mathbf{\Sigma} = \mathbf{\Gamma}\mathbf{\Gamma}^T$

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- Then, $\mathbf{X}$ jointly Normal, and one gets the Gaussian copula

$$\Pr[\{V_i \leqslant v_i\}] = \mathbf{F_X}(\{\theta_i\}) = \mathbf{\Phi}(\{\theta_i\}; \mathbf{\Sigma}) = \mathbf{\Phi}(\{\Phi^{-1}\underbrace{(F_i(v_i))}_{\doteq u_i}\}; \mathbf{\Sigma})$$

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Random Vector X is jointly Gaussian (N=2)

$$
\begin{aligned}
f(x,y) = E_Z[f(x,y|Z)] &= \int \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x-\rho z)^2}{2(1-\rho^2)}} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(y-\rho z)^2}{2(1-\rho^2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \frac{\sqrt{(1+\rho^2)}}{2\pi\sqrt{(1-\rho^4)}} \int \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)}} e^{-\frac{(x-\rho z)^2+(y-\rho z)^2+z^2(1-\rho^2)}{2(1-\rho^2)}} dz \\
&= \varphi(x,y;\rho) e^{\frac{x^2+y^2-2\rho^2 xy}{2(1-\rho^4)}} \int \frac{1}{\sqrt{2\pi}\frac{\sqrt{(1-\rho^2)}}{\sqrt{1+\rho^2}}} e^{-\frac{(x-\rho z)^2+(y-\rho z)^2+z^2(1-\rho^2)}{2(1-\rho^2)}} dz \\
&= \varphi(x,y;\rho) \int \frac{1}{\sqrt{2\pi}\frac{\sqrt{(1-\rho^2)}}{\sqrt{(1+\rho^2)}}} e^{-\frac{(x-\rho z)^2+(y-\rho z)^2+z^2(1-\rho^2)-\frac{x^2+y^2-2\rho^2 xy}{(1+\rho^2)}}{2(1-\rho^2)}} dz \\
&= \varphi(x,y;\rho) \int \frac{1}{\sqrt{2\pi}\frac{\sqrt{1-\rho^2}}{\sqrt{1+\rho^2}}} e^{-\frac{\frac{(x-\rho z)^2+(y-\rho z)^2+z^2(1-\rho^2)}{1+\rho^2}-\frac{x^2+y^2-2\rho^2 xy}{(1+\rho^2)^2}}{2(1-\rho^2)/(1+\rho^2)}} dz \\
&= \varphi(x,y;\rho) \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\frac{(x^2-2\rho xz+y^2-2\rho yz)(1+\rho^2)-(x^2+y^2-2\rho^2 xy)}{(1+\rho^2)^2}+z^2}{2\sigma^2}} dz \\
&= \varphi(x,y;\rho) \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\frac{(x^2+y^2)(1+\rho^2)-(x^2+y^2-2\rho^2 xy)}{(1+\rho^2)^2}-2\rho\frac{x+y)}{1+\rho^2}+z^2}{2\sigma^2}} dz \\
&= \varphi(x,y;\rho) \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{\rho(x+y)}{1+\rho^2}-z\right)^2}{2\sigma^2}} dz
\end{aligned}
$$

Coupling Margins using Gaussian-Gamma Latent Variables

- Assume $V_i \sim F_i$ and CDF $\mathbf{G_V}$ to be estimated
- Consider $X_i \sim T_\nu$ $(\nu > 2)$ and let's Studentize :
  - $\Pr[\{V_i \leqslant v_i\}] = \Pr[\{X_i \leqslant \theta_i\}]$, $\theta_i \doteq T_\nu^{-1}(F_i(v_i))$
  - dependence bw $V_i \Leftrightarrow$ dependence bw X_i
- Assume the following coupling model ($Z_i \sim \Phi$, $\tilde{X}_i \sim$ iid Φ, W s.t. $\nu/W \sim \chi^2_\nu$):

$$X_i = \sqrt{W}\left(\sqrt{1-\Gamma_i\Gamma_i^T}\tilde{X}_i + \Gamma_i\mathbf{Z}^T\right) \Leftrightarrow \tilde{X}_i = \frac{X_i/\sqrt{W} - \Gamma_i\mathbf{Z}^T}{\sqrt{1-\Gamma_i\Gamma_i^T}}$$

- $\Rightarrow$ Conditionally on $(W, \mathbf{Z})$, $X_i \perp X_j \;\; \forall i \neq j$
- $\mathrm{Cov}(X_i, X_j) = \frac{\nu}{\nu-2}\Gamma_i\Gamma_j^T \Rightarrow \mathbf{\Sigma} = \frac{\nu}{\nu-2}\mathbf{\Gamma}\mathbf{\Gamma}^T$

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- $\mathrm{Cov}(X_i, X_j) = \frac{\nu}{\nu-2}\Gamma_i\Gamma_j^T \Rightarrow \mathbf{\Sigma} = \frac{\nu}{\nu-2}\mathbf{\Gamma\Gamma}^T$
- $\mathbf{X}$ is jointly Student-t and one gets the Student-t copula

$$\Pr[\{V_i \leqslant v_i\}] = \mathbf{F_X}(\{\theta_i\}) = \boldsymbol{T}_\nu(\{T_\nu^{-1}(\underbrace{F_i(v_i)}_{u_i})\}; \mathbf{\Sigma})$$

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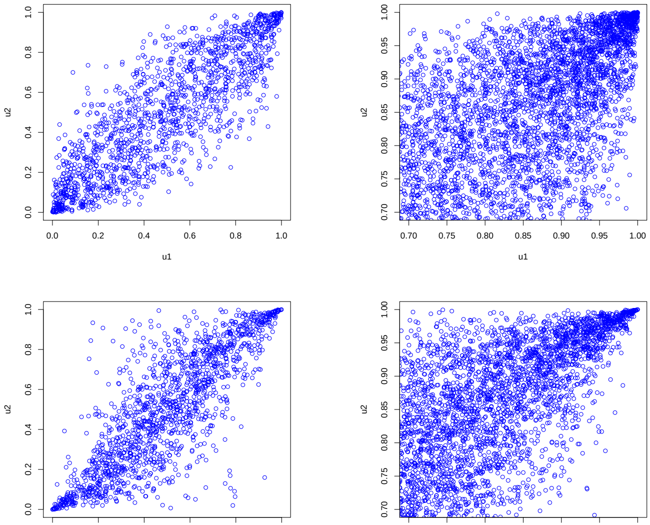
Choose a copula family

- Are your margins independent ? $\Rightarrow$ use $C^{\perp}$ (trivial)
- Are your margins independent in some of the tails ? $\Rightarrow$ do not use $C^{\mathcal{N}}$: "*Regarding of how high a ρ we choose, extreme **tail** events appear to occur independently*"
- Are your margins dependent in the tails even though globally independent ? $\Rightarrow$ use Student "*Regarding of how ρ is chosen close to zero, upper **tail** events appear to be linked*"
- ...

Samples From Gaussian and t-t copulas



parameters estimation

- Full ML
- . . .

Outline

1. Motivation
2. Theoretical Framework
   - Definitions & Key Theorems
   - Other results
   - Examples
   - Copulas and Latent Variable Modeling
3. **Financial application : an example**
   - Credit Default Swap (CDS)
   - Collaterallized Debt Obligation (CDO)
   - Valuing CDOs using copulas

CDS : principles

- is a financial product linked to a *name* (i.e. a company)
- which consists in swapping default losses bw counterparties ($\sim$ insurance contract)
- ex: "*ING* buys 3-year protection on *Ford* to *HSBC* for $N = 10$M Eur"
  - *ING* pays (quarterly) fees $\propto N$ to *HSBC*
  - if *Ford* defaults within 3 years, then *ING* stops paying, *HSBC* pays $(1 - Rec)N$ to ING
- usefull for protecting a loan, an investment or for speculation

$\Rightarrow$ Key quantity : $\Pr[\tau \leqslant t]$ where τ is the default time of *name*.

Outline

CDO : principles

- is a financial product linked to a *basket of names* (indices or bespoke)
- which consists in swapping default losses bw counterparties ($\sim$ insurance contract)
- ex: "*A* buys protection to *B* on the basket against any losses bw *LL* and *UL* (cncy) of basket's notional *N*"
- *A*,*B* are major banks, hedge funds,. . .

CDO : principles

Example: $LL = 3\%N$, $UL = 22\%N$.

- A pays quarterly fees to B, depending on the tranche size ($f \propto (UL - LL)$)
- while Loss(Basket)$\leqslant LL$, f are due.
- if $LL <$ Loss(Basket)$< UL$, B pays $Loss - LL$, A still pay fees $\propto (UL - Loss(Basket))$.
- once Loss(Basket)$\geqslant UL$, fees are not due any longer. B has to pay $(UL - LL)$ to A.
- A,B have different views on how $E[Loss(Basket)]$ will evolve in the future ($\sim$ shares)

CDS/CDO why are they appealing [1/2]

- Allows banks to protect themselves against counterparty risk (loan,. . .)
- Allows banks to give credit loan by transferring credit risk
- Highly leveraged products : interesting for speculation

CDS/CDO why are they appealing [2/2]

Leveraged products :

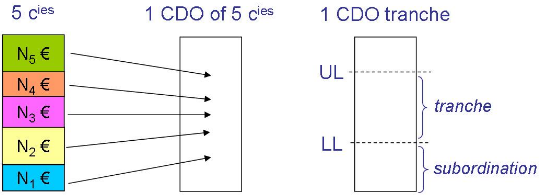
- Assume you want to earn money via shares :
  - Buy x shares at t_0 (expensive: $x * S_{t_0}$), sell at t : cash-in $x(S_t - S(t_0)) - costs$.
  - Your return is typically of the same order as your investment (maybe between .1 $\sim$ 10)
  - you need to put a lot on the table
- Assume you invest in a CDS (sell protection). Then, potentially :
  - Pays nothing (while nothing happens)
  - Cash-in all the periodic payments (fees) !
  - But when default happens. . . gloups.

CDO : price drivers

CDO price depends

- on the risk profile of underlying names (companies)
- on how the names are linked together
- $\Rightarrow$ The key quantity to estimate is $\Pr[\tau_1 \leqslant t, \ldots, \tau_n \leqslant t]$.
- We need to be able to estimate $\Pr[\tau_1 \leqslant t, \ldots, \tau_n \leqslant t]$ from $\Pr[\tau_i \leqslant t]$.

CDO : structuring, tranching and evolution



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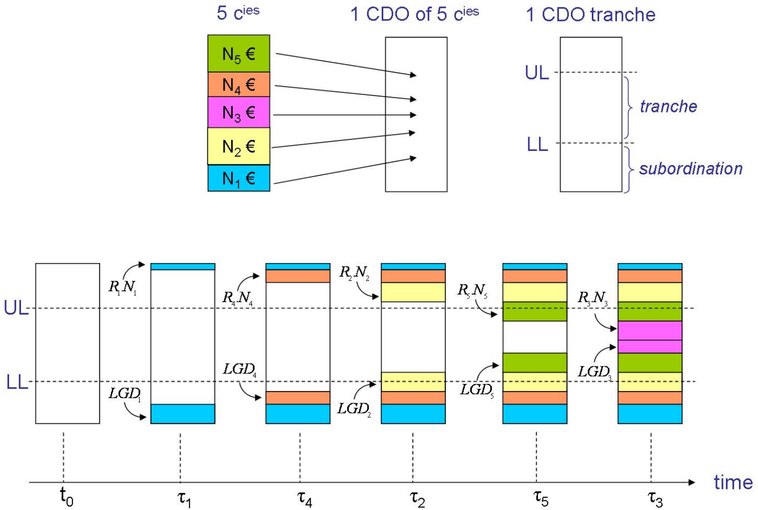

Outline

CDO price is known when so are default times

$$Value \mid \vec{\tau} = \sum_{j=1}^{n} e^{-r(\tau^*_{\pi j}-t_0)} 1_{\{\tau^*_{\pi_j}<T\}} \left\{S_{j-1} \wedge \left(\left(LGD_{\pi_j} - LL_j\right)^+\right)\right\} - s\sum_{j=1}^{n} S_j \int_{t=\tau^*_{\pi j-1}}^{\tau^*_{\pi_j}} e^{-r(\tau^*_{\pi j}-t)} dt$$

$$\vec{\tau} = [\tau_1, \ldots, \tau_n]$$

$$LL_j = \left(LL - \sum_{i=1}^{j-1} LGD_{\pi_i}\right)^+$$

$$UL_j = \left(UL \wedge \left(\sum_{i=1}^{n} N_i - \sum_{i=1}^{j-1} R_{\pi_i} N_{\pi_i}\right)\right)$$

$$S_j = \left(UL_j - \left(K \vee \sum_{i=1}^{j} LGD_{\pi_i}\right)\right)^+$$

$$\pi_1 = \arg\min_i \tau_i \,,\; \pi_2 = \arg\min_{i \neq \pi_1} \tau_i \,, \ldots$$

$$\tau_i^* = T \wedge \tau_i$$

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CDO price as a function of default time distribution

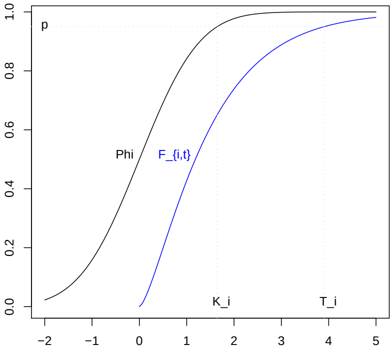
$$Value = E_{\vec{\tau}}\left[Value \mid \vec{\tau}\right] = \int_{\vec{t}} \underbrace{\left(Value \mid \vec{\tau} = \vec{t}\right)}_{known} dF_{\vec{\tau}}(\vec{t})$$

$$F_{\vec{\tau}}(\vec{t}) = \Pr\left[\tau_1 \leq t_1, \ldots, \tau_n \leq t_n\right]$$

Gaussianization of RVs...

- Assume $\tau_i \sim F_i$, $X_i \sim \Phi$ and $K_i \doteq \Phi^{-1}(F_i(T_i))$:

  $\Pr[\tau_i \leqslant T_i] = F_i(T_i) = \Phi(\Phi^{-1}(F_i(T_i))) = \Phi(K_i) = \Pr[X_i \leqslant K_i]$



. . . Leads to (1 Factor) Gaussian copula

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- Define $X_i \doteq \rho Z + \sqrt{1-\rho^2}\tilde{X}_i$ with $Z \perp \tilde{X}_i \sim \Phi$ i.i.d.

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- Then, $\mathbf{X} \sim \mathbf{\Phi}(.; \mathbf{\Sigma})$ where $\Sigma_{ij} = \delta_{ij} + (1-\delta_{ij})\rho^2$

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- Then, $\mathbf{X} \sim \mathbf{\Phi}(.;\mathbf{\Sigma})$ where $\Sigma_{ij} = \delta_{ij} + (1-\delta_{ij})\rho^2$
- Then:

$$\Pr[\{\tau_i \leqslant T_i\}] = \Pr[\{X_i \leqslant K_i\}] = C_{\mathcal{N}}(\{\Phi^{-1}(F_i(T_i))\})$$

Practical Computation of the joint CDF

- Marginal Conditional Probability

$$\Pr[X_i \leqslant \Phi^{-1}(F_i(T_i)) | Z = z] = \Pr\left[\tilde{X}_i \leqslant \frac{\Phi^{-1}(F_i(T_i)) - \rho z}{\sqrt{1-\rho^2}}\right]$$

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- Joint Conditional Probability:

$$\Pr[\{X_i \leqslant \Phi^{-1}(F_i(T_i))\} | Z = z] = \prod_i \Pr\left[\tilde{X}_i \leqslant \frac{\Phi^{-1}(F_i(T_i)) - \rho z}{\sqrt{1-\rho^2}}\right]$$

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- $\Pr[\{\tau_i \leqslant T_i\}] = \Pr[\{X_i \leqslant K_i\}] = E_Z[\Pr[\{X_i \leqslant K_i\} | Z]] =$

$$\int_z \prod_i \phi\left[\frac{K_i - \rho z}{\sqrt{1-\rho^2}}\right] \varphi(z) dz \approx \sum_j w_j \prod_i \phi\left[\frac{K_i - \rho z_j}{\sqrt{1-\rho^2}}\right]$$

Extensions

- From 1 to K Factors : $\boldsymbol{\rho}, \mathbf{Z} \in \mathbb{R}^K$:

$$X_i(t) = \sqrt{1 - \boldsymbol{\rho}\boldsymbol{\rho}^T}\tilde{X}_i + \boldsymbol{\rho}\mathbf{Z}^T \Rightarrow \Sigma_{ij} = \delta_{ij} + (1 - \delta_{ij})\boldsymbol{\rho}\boldsymbol{\rho}^{\mathbf{T}}$$

and $\Pr[\{\tau_i(t) \leqslant T_i\}]$ is given by

$$\int_{\mathbb{R}^K} \prod_i \phi\left[\frac{\phi^{-1}(F_i(T_i)) - \boldsymbol{\rho}\mathbf{z}^T}{\sqrt{1 - \boldsymbol{\rho}\boldsymbol{\rho}^T}}\right] \prod_{k=1}^{K} \varphi(z_k) dz_k$$

- Non-Gaussian couplings (X_k identical copies of X):

$$\varphi_X(\boldsymbol{\theta}) = \prod_{k=1}^{K} \varphi_{X_k}(\boldsymbol{\theta}_k) \quad \textit{Lévy}\text{'s } \infty\text{-divisibility}$$

where $X = \sum_{k=1}^{K} X_k$, $X \sim f(\boldsymbol{\theta})$, $X_k \sim f(\boldsymbol{\theta}_k)$, $X_i \perp X_{j \neq i}$

Summary

- Copulas is a nice theoretical framework for multidimensional CDF estimation
- Copulas are useful for measuring distance and dependence between RVs
- Copulas are used for modeling coupling schemes of real-world applications like finance, biostatistics and psychology

- Theoretical challenges
  - An alternative to the lower bound C^m that would belong to $\mathcal{C}(N)$
  - Enhance statistical inference (copula estimation)

For Further Reading

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  Stat. and Prob. Lett. 63:157-163, 2003.

ING Disclaimer

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