Analytical Credit Basket Pricing

with automatic calibration to CDS curves

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Context: Credit-Based Financial Instruments

- Valuation of financial products: price = $f(\vec{\tau})$, vector of $N \geqslant 1$ default times $\vec{\tau} \doteq (\tau_1, \ldots, \tau_N)$
- Examples : CDS, CDO, FtD, NtD,...
- We need a tractable default model for the joint CDF F of $\vec{\tau}$

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- We need a tractable default model for the joint CDF F of $\vec{\tau}$
  - speed (computationally attractive)
  - flexible (calibration capabilities)
  - sparse (not too many parameters)
- Two-step (bottom-up) approach to create a multivariate model:
  1. model the univariate distributions $F_i(x) \doteq \Pr[\tau_i \leqslant t]$
  2. couple the F_i's to create the joint distribution F

Outline

1. **Model Setup**
   - Univariate Default Model (margins)
   - Multivariate Default Model
2. **Copula Models**
   - Static Copula Models
   - Dynamic Copula Models
3. **Recent Jump Models**
   - Mai & Scherer Default Model
   - Hull & White Default Model
   - First-to-Default Swaps
   - Other Applications
4. **Conclusion**

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Default Model ($N = 1$): Standard set up

- Intensity process : $\lambda_i(t) > 0$ ($\forall t > 0$)
- Probabilities : let $\Lambda_i(t) \doteq \int_{s=0}^{t} \lambda_i(s)ds$, then

$$S_i(t) \doteq \Pr[\tau_i > t] = e^{-\int_{s=0}^{t} \lambda_i(s)ds} = e^{-\Lambda_i(t)}$$

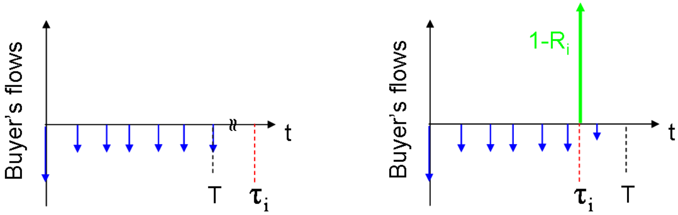
$$F_i(t) \doteq \Pr[\tau_i \leqslant t] = 1 - S_i(t)$$

- Meaning :
  - $\lambda_i(t) \sim$ default rate @ t (= $\lim_{\Delta \to 0} \Pr[\tau_i \leqslant t + \Delta | \tau_i > t]/\Delta$)
  - $\lambda_i(t) \sim$ deterministic : piecewise constant bw tenors
  - $\tau_i \sim 1^{st}$ jump of Poisson process with intensity $\lambda_i(t)$
- How to find $\lambda_i(t)$?

Default Model ($N = 1$): Calibration of intensities on CDS quotes

- Protection buyer pays upfront + premium up to maturity or default, whichever comes first
- Protection seller pays non-recovered part of notional if default comes prior to maturity

Scenario 1 ($\tau_i > T$)   *Scenario 2* ($\tau_i < T$)



CDS : Pricing equations

Pricing of Protection leg (CL) and Premium leg (FL) :

$$
\begin{aligned}
CL &\doteq \mathbb{E}\left[(1-R)\delta(\tau_i)\,\mathbb{1}_{\{\tau_i \leqslant T\}}\right] = (1-R)\int_{t=0}^{T}\delta(t)f_i(t)dt \\
FL &\doteq \mathrm{up} + s\sum_{k=1}^{K}\delta(t_k)\mathbb{E}\left[t_k \wedge (t_{k-1} \vee \tau_i) - t_{k-1}\right] \\
&= s\sum_{k=1}^{K}\delta(t_k)\left((t_k - t_{k-1})S_i(t_k) - \int_{t=t_{k-1}}^{t_k}(t - t_{k-1})dS_i(t)\right)
\end{aligned}
$$

with $\delta(t)$ (disc. fact. at t), s (spread), $\{t_k\}$ (payment dates) and T (maturity)

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Multivariate : N underlying entities are gathered in a portfolio

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- Information about coupling is lacking

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  1. with random variables (static: e.g. factor-copulae)
  2. with random processes (dynamic)

Why is a Model Needed ?

Market quotes give prices

- A model not in line with market is useless
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Well Ok, but you need

- To value your positions on a daily basis (P& L)
- To value *side products* (e.g. infer “market implied correlations”)
- To compute sensitivities to risk factors

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Static Latent Variable Models

- Let $X_1, \ldots, X_N$ be dependent RVs, $X_i \sim F_{X_i}$
- Assume each has an idiosynchratic & systemic part

$$
\begin{aligned}
X_i &\doteq \alpha_i \tilde{X}_i + \mathbf{\Gamma}_i \mathbf{Z}^T \\
X_j &\doteq \alpha_j \tilde{X}_j + \mathbf{\Gamma}_j \mathbf{Z}^T
\end{aligned}
$$

with $\mathbf{\Gamma}_{i,j} \in \mathbb{R}^K$, Z_i i.i.d., $Z_i \perp \tilde{X}_j$, $\tilde{X}_i \perp \tilde{X}_{j \neq i}$, $\mathrm{Cov}(X_i, X_j) = \mathbf{\Gamma}_i \mathbf{\Gamma}_j^T$

- Choosing the latent variables $\Leftrightarrow$ choosing the copula

Calibration of Copula Models

- Define the time-dependent barrier $K_i(t) \doteq F_{X_i}^{[-1]}(F_i(t))$:

$$\begin{array}{rcl} \Pr[X_i \leqslant K_i(t)] & = & \Pr[\tau_i \leqslant t] \\ & \Downarrow & \\ \mathbb{E}\left[1\!\!1_{\{X_i \leqslant K_i(t)\}}\right] & = & \mathbb{E}\left[1\!\!1_{\{\tau_i \leqslant t\}}\right] \end{array}$$

- We can work with X_i instead of τ_i
- X_i are correlated credit worthiness RVs, from which joint probabilities can be computed

Examples of One Factor Copulas

- Gaussian copula $Z \sim \tilde{X}_i \sim \mathcal{N}(0,1), \alpha_i = \rho, \gamma_i = \sqrt{1-\rho^2}$
- $t - t(\nu)$ copula $Z \sim \tilde{X}_i \sim t_\nu, \alpha_i = \rho\sqrt{\frac{\nu-2}{\nu}}$,
  $\gamma_i = \sqrt{1-\rho^2}\sqrt{\frac{\nu-2}{\nu}}$
- Gamma copula $Z \sim \Gamma(x; \phi\gamma, \theta)$, $\tilde{X}_i \sim \Gamma(x; (1-\phi)\gamma, \theta)$
  $\alpha_i = \gamma_i = 1, \text{Corr}(X_i, X_j) = \phi$

Drawbacks of Static Copula Models

- Coupling is not time-dependent
- *Systemic variable* Z is not time-dependent (likelihood for the economy state is frozen)
- wouldn't not be possible to introduce dynamics ?

Outline

Dynamic Latent Process Models

- Let $X_1(t), \ldots, X_N(t)$ be dependent RPs, $X_i(t) \sim F_{X_i(t)}$
- Assume each has an idiosynchratic & systemic part

$$X_i(t) \doteq \alpha_i \tilde{X}_i(t) + \mathbf{\Gamma}_i \mathbf{Z}^T(t)$$
$$X_j(t) \doteq \alpha_j \tilde{X}_j(t) + \mathbf{\Gamma}_j \mathbf{Z}^T(t)$$

with $\mathbf{\Gamma}_{i,j} \in \mathbb{R}^K$, $Z_i(t)$ i.i.d., $Z_i(t) \perp \tilde{X}_j(t)$, $\tilde{X}_i(t) \perp \tilde{X}_{j \neq i}(t)$

- Choosing the latent processes $\Leftrightarrow$ choosing the dynamic coupling

Dynamic Copula Models : Calibration to CDS

- Random *variables* X_i becomes random processes $X_i(t)$
- Modeling default events and probabilities

$$\tau_i = \inf\{t \geqslant 0 : X_i(t) > K_i(t)\}$$

$$\Downarrow$$

$$\Pr[\tau_i \leqslant t] = \Pr\left[\max_{s \leqslant t}\Big(X_i(s) - K_i(s)\Big) > 0\right]$$

Dynamic Copula : Monte Carlo Pricing

1. Given latent processes dynamics, calibrate barrier $K_i(t)$
2. For each Monte Carlo trial :
   - Generate latent processes for $t \in \mathcal{T} \doteq \{0, \Delta t, \ldots, T\}$: $\tilde{X}_i(t)$, $Z(t)$ yielding that of the default processes $X_i(t)$
   - Look for 1^{st} passage times of $X_i(t)$ above $K_i(t)$ before maturity
   - Given the above, price swaps with payoff model
3. Average figures to obtain final price

Lévy Processes in One Slide

$X(t)$ is a Lévy process if it is a stochastic process s.t.

- starts at 0 : $X(0) = 0$ almost surely (a.s.)
- has independent increments :
  $X(t_2) - X(t_1) \perp X(t_4) - X(t_3)$ for $t_1 < t_2 \leqslant t_3 < t_4$
- has stationary increments : $X(t + \Delta) - X(t) \sim X(\Delta)$ for all $t > 0$ and any $\Delta > 0$
- is càdlag

Why Lévy Processes ?

- Let $X(t)$ be a Lévy process and define $X = X(1)$. Then :

$$\phi_{X(t)}(u) = \phi_X^t(u) \qquad (\phi_X(u) \doteq \mathbb{E}[e^{iuX}]), \quad i \doteq \sqrt{-1}$$

- Consequently, because *characteristic function* satisfies

$$\phi_{X+Y}(u) \overset{X \perp Y}{=} \phi_X(u)\phi_Y(u)$$

we know the CF $\phi_{X_i(t)}$ based on $\phi_{Z(1)}, \phi_{\tilde{X}_i(1)}$, and t

Dynamic Gamma Model : Equations

- $X_i(t) = \tilde{X}_i(t) + Z(t)$
- $\tilde{X}_i(t) \sim \mathbf{\Gamma}(x; (1-\phi)\gamma t, \theta)$ and $Z(t) \sim \mathbf{\Gamma}(x; \phi\gamma t, \theta)$, $\theta > 0$
- Hence, characteristic function of $X_i(t)$ equals

$$\phi_{X_i(t)}(u) = (1-\theta iu)^{-(1-\phi)\gamma t}(1-\theta iu)^{-\phi\gamma t} = (1-\theta iu)^{-\gamma t}$$

and by identification, one gets :

$$X_i(t) \sim \mathbf{\Gamma}(x; \gamma t, \theta)$$

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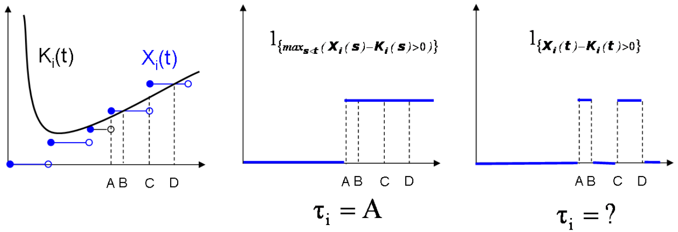
Dynamic Gamma Model : Not so nice

- "max" is needed :



$\Rightarrow$ calibration to CDS is intensive

- For each change of (γ, θ), CDS calibration is needed (CDS cal insensitive to ϕ)
- Pricing & calibration requires Monte Carlo !

Outline

Mai & Scherer Default Model [2009]

Model the default indicator $A_i(t) \doteq 1\!\!1_{\{\tau_i \leqslant t\}}$ as

$$A_i(t) = 1\!\!1_{\{E_i \leqslant \xi_{f_i(t)}\}}$$

where

- ξ_t is an *almost surely positive* Lévy process (*subordinator*)
- E_i are i.i.d. random barriers (random, but time-independent thresholds)
- $f_i(t) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is some grounded increasing functional

Mai & Scherer : Time distorsion function

Default model :

$$A_i(t) = \mathbb{1}_{\{E_i \leqslant \xi_{f_i(t)}\}}$$

where

- $\Rightarrow \tau_i$ is modeled as 1^{st} passage time of stochastic process $\xi_{f_i(t)}$ above a random (but time-independent) barrier E_i
- $f_i(t) \doteq \Lambda_i(t)$ (obtained from CDS)
- If $X(t)$ increasing and $K(t) = K$ then $\Pr\left[\max_{s \leqslant t}\left(X(s) - K(s)\right) > 0\right] = \Pr\left[X(t) > K\right]$
- Would be much easier !

Lévy subordinator : increasigness and Laplace exponent

A Lévy subordinator ξ_t is

- a.s. increasing
- uniquely defined via its Laplace exponent $\Psi(s)$:

$$\mathbb{E}[e^{s\xi_t}] = e^{t\Psi(s)}$$

Connexion of Laplace exponent with characteristic function :

$$\Psi(s) = \log(\phi_{\xi_1}(-is)) \; , \quad \phi_{\xi_1}(u) \doteq \mathbb{E}[e^{-iu\xi_1}]$$

Mai & Scherer : Calibration to CDS Survival Probabilities

Survival probability of entity i as given by model :

$$\begin{aligned}\Pr[E_i > \xi_{f_i(t)}] &\overset{E_i \sim Exp(1)}{=} \mathbb{E}[e^{-\xi_{f_i(t)}}] \\ &= e^{f_i(t)\Psi(-1)}\end{aligned}$$

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So, calibration to CDS probs is achieved provided that

$$\Psi(-1) = -1$$

Mai & Scherer : Examples of Subordinators

- Compound Poisson process : $\xi_t = \mu t + \sum_{j=1}^{J_t} \tilde{E}_j$, J_t homogeneous Poisson process with rate β and $\tilde{E}_j \sim Exp(\eta)$

$$\Psi(s) = \mu s + \beta s/(\eta - s) \Leftrightarrow \mu = 1 - \beta/(\eta + 1)$$

- Gamma process : $\xi_t = \mu t + \Gamma_t$, $\Gamma_t \sim \gamma(at, b)$

$$\Psi(s) = \mu s + a \log(b/(b - s)) \Leftrightarrow \mu = 1 + a \log(b/(b + 1))$$

- Inverse Gaussian : $\xi_t = \mu t + IG_t$, $IG_t \sim \mathbf{IG}(at, b)$

$$\Psi(s) = \mu s - a(\sqrt{b^2 - 2s} - b) \Leftrightarrow \mu = 1 - a(\sqrt{b^2 + 2} - b)$$

Mai & Scherer : Survival function

Let Ψ be the Laplace exponent of the chosen subordinator. Then

$$\begin{aligned} S(t_1,\ldots,t_N) &\doteq \Pr[\tau_1 > t_1,\ldots,\tau_N > t_N] \\ &= S_{(1)}(t_1)\prod_{j=2}^{N} S_{(j)}(t_j)^{\Psi(-(j-1))-\Psi(-j)} \end{aligned}$$

where $S_{(1)}(t_1) \leqslant S_{(2)}(t_2) \leqslant \ldots \leqslant S_{(N)}(t_N)$ is the ordered list of $\{S_1(t_1),\ldots,S_N(t_N)\}$

Mai & Scherer : First to default distribution

Let $\tau^{(1)} \doteq \min_j \tau_j$ (RV):

$$
\begin{aligned}
S(t) &\doteq \Pr[\tau^{(1)} > t] \\
&= S(t, \ldots, t) \\
&= S_{(1)}(t) \prod_{j=2}^{N} S_{(j)}(t)^{\Psi(-(j-1))-\Psi(-j)}
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Outline

Modeling Intensities : Hull & White [1/2]

- Link bw entities : $\Lambda_i(t)$ become stochastic: $\Lambda_i(t) \Rightarrow \tilde{\Lambda}_i(t)$
- $\tau_i \sim 1^{st}$ jump of Cox process
- $S_i(t) = \Pr[P_i(t) > U_i]$, $U_1, \ldots, U_N$ are $\mathcal{U}(0,1)$ rv

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Examples: ($U_i \perp U_j$ and $P_i(t) = e^{-\tilde{\Lambda}_i(t)}$)

$$\tilde{\Lambda}_i(t) \overset{Mai-Scherer}{=} \xi \circ \Lambda_i(t), \ \xi_t = \text{Lévy subordinator}$$

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$$\overset{Hull-White}{=} \int_{s=0}^{t} \mu_i(s)ds + \sum_{j=1}^{J_t} H(j)$$

- $M_i(t) \doteq \int_{s=0}^{t} \mu_i(s)ds$ is a cumulative deterministic drift
- J_t is a inhomogeneous Poisson process with intensity $\lambda(t)$
- $H(j) > 0$ defines size of j^{th} jump of $\sum_{j=1}^{J_t} H(j)$

This setup also includes the static copula approach

- Let $\Lambda_i(t)$ be deterministic cumulative intensities
- Let $U_i \doteq \Phi(-X_i)$, $U_j \doteq \Phi(-X_j)$, $X_i = \rho Z + \sqrt{1-\rho}\tilde{X}_i$
- So, U_i linked to U_j via (one-factor Gaussian) copula:

$$
\begin{aligned}
S(\vec{t}) &= \Pr\left[\bigcup_{i=1}^{n}\left\{e^{-\Lambda_i(t_i)} > U_i\right\}\right] \\
&= \Pr\left[\bigcup_{i=1}^{n}\left\{\Phi^{-1}[S_i(t_i)] > -X_i\right\}\right] \\
&= \mathbb{E}\left[\Pr\left[\bigcup_{i=1}^{n}\left\{\frac{\Phi^{-1}[F_i(t_i)] - \rho Z}{\sqrt{1-\rho}} < \tilde{X}_i\right\} \middle| Z\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{n}\left(1 - \Phi\left[\frac{K_i(t_i) - \rho Z}{\sqrt{1-\rho}}\right]\right)\right]
\end{aligned}
$$

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Modeling Intensities : Hull & White [2/2]

- Problem : no analytical results worked out
- brute force : calibration to CDS,...
- we propose to focus on $H(j) = H$ (constant jump), and work out key results

Hull & White : Calibration to CDS

Survival probability of entity i as given by model:

$$\begin{aligned}
\mathbb{E}\left[e^{-M_i(t)-\sum_{j=1}^{J_t} H(j)}\right] &= e^{-M_i(t)}\mathbb{E}[e^{-J_t H}] \\
&= e^{-M_i(t)}\phi_{J_t}(iH) \\
&\overset{\Lambda(t)\doteq\int_{s=0}^{t}\lambda(s)ds}{=} e^{-M_i(t)}e^{\Lambda(t)(e^{-H}-1)}
\end{aligned}$$

So, calibration to CDS probs is achieved up to t provided that

$$\mu_i(s) \overset{\forall s\leqslant t}{=} \lambda_i(s)+\lambda(s)(e^{-H}-1) \Leftrightarrow \left\{\Psi(-1)=-1,\ f_i(s) \overset{\forall s\leqslant t}{=} \Lambda_i(s)\right\}$$

Modeling probabilities vs Modeling events

- So far, we have required $\Pr[e^{-\tilde{\Lambda}_i(t)} > U_i] = \Pr[\tau_i > t]$
- This is not the same as requiring $\{\tau_i > t\} = \left\{e^{-\tilde{\Lambda}_i(t)} > U_i\right\}$
- Proper modelization requires modeling events

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- Ex: if we want to model $1\!\!1_{\{\tau_i > t\}}$ via $1\!\!1_{\left\{e^{-\tilde{\Lambda}_i(t)} > U_i\right\}}$ we need to further require that $\tilde{\Lambda}_i(t)$ a.s. increasing

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- Therefore, we need the additional constraint $\mu_i(s) > 0$, which defines a range for $(\lambda(s), H)$.
- Condition $\mu_i(s) > 0$ is not needed to fit $S_i(t)$, but necessary to get proper conditional and joint distributions.

Hull & White : Jointure function

Let $\psi(N, H, \Lambda(t))$ be the jointure function :

$$\psi(N, H, \Lambda(t)) \doteq e^{\Lambda(t)\left((e^{-NH}-1)-N(e^{-H}-1)\right)}$$

It holds that

$$\psi(N, 0, \Lambda(t)) = \psi(N, H, 0) = 1$$

and if $N > 1, H > 0, \Lambda > 0$, then

$$\psi(N, H, \Lambda) > 1$$

Hull & White : Survival function

If $H(j) = H$ and $\lambda(t) = \lambda$:

$$\begin{aligned} S(t_1, \dots, t_N) &\doteq \Pr[\tau_1 > t_1, \dots, \tau_N > t_N] \\ &= \prod_{i=1}^{N} S_i(t_i)\psi\Big(N - i + 1, H, (t_{(i)} - t_{(i-1)})\lambda\Big) \\ &= S^{\perp}(\vec{t}) \prod_{i=1}^{N} \psi\Big(N - i + 1, H, (t_{(i)} - t_{(i-1)})\lambda\Big) \end{aligned}$$

where $0 = t_{(0)} \leqslant t_{(1)} \leqslant \dots \leqslant t_{(N)}$ is the ordered list of $\{t_1, \dots, t_N\}$ and $S^{\perp}(\vec{t}) \doteq \prod_{i=1}^{N} S_i(t_i)$

Hull & White : First to default distribution

Let $\tau^{(1)} \doteq \min_j \tau_j$:

$$\begin{aligned} S(t) &\doteq \Pr[\tau^{(1)} > t] \\ &= S(t, \ldots, t) \\ &= \psi\Big(N, H, \Lambda(t)\Big) S^{\perp}(t) \end{aligned}$$

where

$$S^{\perp}(t) \doteq \prod_{i=1}^{N} S_i(t)$$

Impact of jointure function ($N = 2$)

- Because $\psi \geqslant 1$: $PQD \Rightarrow \rho \geqslant 0$
- Pearson's correlation coefficient of $A_i(t) \doteq 1\!\!1_{\{\tau_i \leqslant t\}}$:

$$\mathrm{Corr}(A_i(t), A_j(t)) = \rho_{ij}(t) = (\psi(2, H, \Lambda(t)) - 1)\sqrt{f(S_i(t), S_j(t))}$$

- Short-term default correlation $\rho_{ij}(0) \doteq \lim_{t \downarrow 0} \rho_{ij}(t)$:

$$\rho_{ij}(0) = \frac{\log \psi(2, H, \lambda(0^+))}{\sqrt{\lambda_i(0^+)\lambda_j(0^+)}} \quad (\text{for GC :} \rho_{ij}(0) = 0)$$

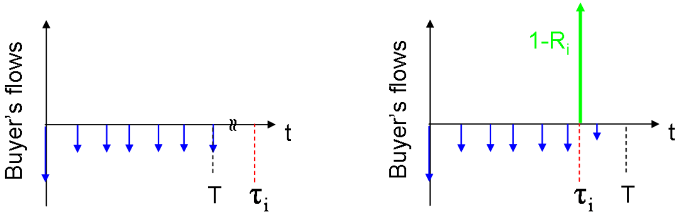
ING

Outline

FtD : Principle close to a CDS

Same as CDS except that τ_i is replaced by $\tau^{(1)} \doteq \min_{i\in\{1:N\}} \tau_i$

- Protection buyer pays upfront + premium up to maturity or first default, whichever comes first
- Protection seller pays non-recovered part of notional if first default comes prior to maturity



FtD & CDS : why so (un)popular instruments

They are very useful instruments

- Insurance against default
- Allow financial institutions to give easier access to credit
- Contract can be sold and bought: high liquidity

FtD & CDS : why so (un)popular instruments

They are very useful instruments

- Insurance against default
- Allow financial institutions to give easier access to credit
- Contract can be sold and bought: high liquidity

However:

- Insurance on someone else's asset
- No need to have exposure to buy/sell credit protection
- Can be used to speculate on company's defaults
- Defaults are not needed to make money (spread widening)

FtD & CDS : why so (un)popular instruments

Evolution of Historical 5Y CDS spread of

- Belgium ($\times$ 10 in 2.5Y)
- Greece ($\times$ 10 in 6M)

Regulatory rules:

- Only for financial institutions (you forget it)
- In the future, only available to hedge real exposures ?
- What about liquidity then ?

FtD : pricing as CDS pricing

- In terms of $S(t) \doteq \Pr[\tau^{(1)} > t]$, with $R_i = R$:

$$\begin{aligned} CL &\doteq -(1-R)\int_{t=0}^{T} \delta(t)dS(t) \\ FL &\doteq s\sum_{k=1}^{K} \delta(t_k)\int_{t=t_{k-1}}^{t_k} S(t)dt \end{aligned}$$

- Just like a CDS but where $S(t)$ is not $\Pr[\tau_i > t]$ but $\Pr[\min_i \tau_i > t]$

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Hull & White : FtD priced as a CDS with intensity

$\Rightarrow$ If $R_i = R$, FtD = CDS :

$$S(t) = e^{-\tilde{\Lambda}(t)}, \quad \tilde{\Lambda}(t) \doteq \sum_{i=1}^{N} \Lambda_i(t) - \underbrace{\log \psi(N, H, \Lambda(t))}_{\doteq \Lambda_0(t)}$$

$\Rightarrow$ FtD could be priced & calibrated with a HR-CDS pricer

$\Rightarrow$ FtD price range :

- $\overline{sp}$ (highest price): $\Lambda_0(t) = 0$ (independence)
- $\underline{sp}$ (smallest price): $\Lambda_0(t) = \sum_{i=1}^{N} \Lambda_i(t)$? ($\tilde{\lambda} = 0 \Rightarrow sp = 0$)
- Actually, $\underline{sp} > 0$ as one must have $\Lambda_0(t) < \sum_{i=1}^{N} \Lambda_i(t)$

Handling $R_i \neq R_j$ cases [1/2]

- Usually, CL modifies to

$$\int_{t=0}^{T} \delta(t) \sum_{i=1}^{N} (1-R_i) \underbrace{\lim_{\Delta \to 0} \Pr\left[\tau_i \in [t, t+\Delta), \sum_{j \neq i} A_j(t) = 0\right] / \Delta \, dt}_{\doteq f_{i,1}(t)}$$

  where $f_{i,1}(t)$ is the density for i to cause the 1^{st} default
- This works only when $\Pr[\tau_i = \tau_j] = 0$ (ok for continuously differentiable copulae)

Handling $R_i \neq R_j$ cases [2/2]

- If correlation results from jumps, this is not correct
- This is because simultaneous defaults are allowed
- In general (i.e. when $S(t,\dots,t) \neq S^{\perp}(t)$ and $n > 1$)

$$\sum_{i=1}^{N} f_{i,1}(t) - f^{(1)}(t) \overset{H\&W}{=} \log\Big(\psi(N,H,\lambda(t))\Big) S(t) \neq 0$$

- Why ? $\Rightarrow$ "double-countings" (H&W : $\psi(N,H,\lambda(t)) > 1$)
- In H&W model, we can come up with a workaround

Handling $R_i \neq R_j$ cases in H& W [1/2]

- Split cases where $\tau^{(1)}$ results from isolated or simultaneous defaults

$$\underbrace{\Pr[\tau^{(1)} \in [t, t+ds)]}_{(A)} = \underbrace{\Pr[\tau^{(1)} \in [t, t+ds), dN(\tau^{(1)}) = 1]}_{(B)} + \underbrace{\Pr[\tau^{(1)} \in [t, t+ds), dN(\tau^{(1)}) > 1]}_{(C)}$$

- Term (C) is obtained by (A)-(B) and :

$$(B) = \sum_{i=1}^{N} \Pr[\tau_i \in [t, t+ds), dN(\tau_i) = 1]$$

Handling $R_i \neq R_j$ cases in H& W [2/2]

- For cases where no simultaneous default happen (B), we use the associated recovery (no ambiguity)
- For the other cases, we use $\hat{R} \doteq g(R_1, \ldots, R_N)$ (e.g. max,min,mean,...)

$$\Big\{\sum_{i=1}^{N}(1-R_i)\left(\lambda_i+\log\left(\frac{\psi(N-1,H,\lambda)}{\psi(N,H,\lambda)}\right)\right)+(1-\hat{R})$$
$$\times\Big((N-1)\log(\psi(N,H,\lambda))-N\log(\psi(N-1,H,\lambda))\Big)\Big\}$$
$$\times\int_{t=0}^{T}\delta(t)S(t)dt$$

- If $g(R,\ldots,R)=R$, works for homogeneous pools too

Playing with (H, λ) = playing with tails [1/2]

Assume :

- $r(t) = 0$ (or equivalently, $\delta(t) = 1$, ie no interest rates)
- $S_i(t) = e^{-\lambda_i t}$ (one-tenor or one avg intensity up to maturity)
- $R_i = R$ (homogeneous recoveries)

Then, the iso-FtD curve $(H, \lambda(H, sp^\star))$ yielding a same fair spread $s^\star$ for FtD is given by

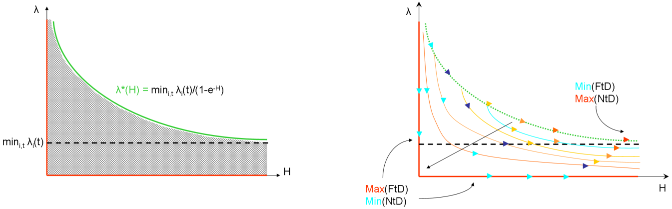
$$\lambda(H, s^\star) = \frac{\sum_{i=1}^{N} \lambda_i - \frac{sp^\star}{(1-R)}}{\log \psi(N, H, 1)}, \quad (H > 0)$$

Indeed, in that case $\tilde{\lambda} = \lambda^\star$ where $\lambda^\star \doteq \frac{s^\star}{(1-R)}$ is the "fair intensity", ie the intensity such that CDS has a zero MtM when priced with $s^\star$ (when $r(t) = 0$).

$\Rightarrow$ Handy to calibrate KtD given FtD

Playing with (H, λ) = playing with tails [2/2]

- Couples (H, λ) fitting a same FtD price (= survival curve)



- Increase H s.t. FtD price is constant means
  - increase probability of catastrophe scenario
  - decrease implied jump intensity λ

Outline

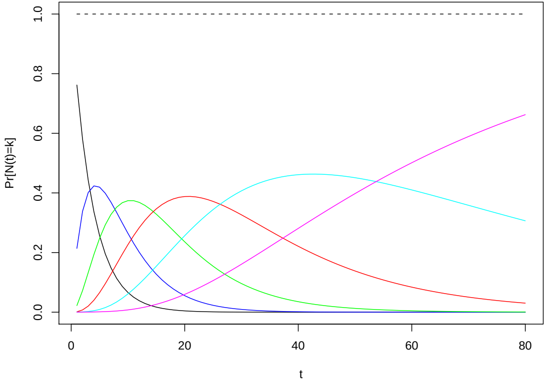
Hull & White : k^{th}-to-Default pricing [1/2]

- $\Pr[N(t) = k]$ is tractable for medium baskets
- Combinatorial but analytically tractable
- No approximation, no numerical integration, no recursion

$$\Pr[N(t) = k] = \sum_{\substack{1 \leqslant i_1 < \cdot < i_k \leqslant N \\ \{i_1, \cdot, i_k, j_1, \cdot, j_{N-k}\} = \{1, \cdot, N\}}} \prod_{k'=1}^{N-k} S_{j_{k'}}(t) \times \Big\{ \psi(N-k) + \sum_{l=1}^{k} (-1)^l \psi(N+l-k) \sum_{1 \leqslant m_1 < \cdot < m_l \leqslant k} \prod_{z=1}^{l} S_{i_{m_z}}(t) \Big\}$$

Hull & White : kth to default pricing [2/2]

- Example : N=5



Pricing CDO tranches

- CDO pricing: requires loss distribution at t

$$\Pr[L(t) = l] = \Pr\left[\sum_i \omega_i \, 1\!\!1_{\{\tau_i \leqslant t\}} = l\right]$$

$$= \sum_{j=1}^{\infty} \Pr\left[\sum_i \omega_i \, 1\!\!1_{\{e^{-M_i(t)-jH} < U_i\}} = l\right] \Pr[J(t) = j]$$

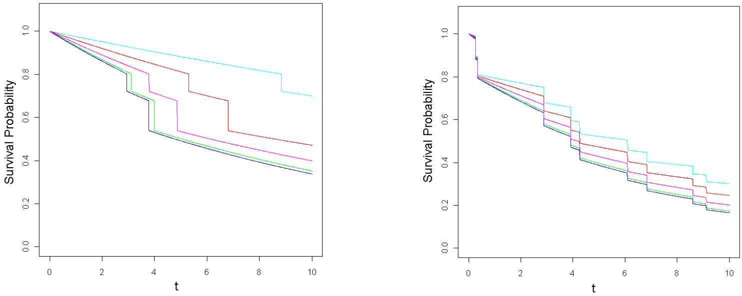
- $B_i(j) \doteq 1\!\!1_{\{e^{-M_i(t)-jH} < U_i\}}$ are independent Bernoulli random variables with probability of success $p_i \doteq 1 - e^{-M_i(t)-jH}$.
- CF of $\sum_i \omega_i B_i(j)$ at u is $\prod_i \phi^{B_i(j)}(u\omega_i) = \prod_i (q_i + p_i e^{\sqrt{-1}u\omega_i})$
- $\Pr[L(t) \leqslant l | J(t) = j]$ obtain by inversion of CF via Gil-Pelaez Theorem ($\mathcal{FFT}^{-1}$)

Outline

Summary : Dynamic Models

- Standard copula models are static
- Dynamic copula difficult to work out (time-dependent barrier : no closed from solution for CDS calibration,...)
- Idea : Modeling multi-dimensional “intensity” processes with jumps to obtain sufficiently high default correlation
- Recent examples: Mai & Scherer, Hull & White

Stochastic Survival Probabilities Given $(J_s)_{s\leqslant t}$



- Left (Mai & Scherer) : survival prob. are time-stretched copies
- Right (Hull & White) : survival prob. jump together by $-H$

Summary : Jump models

- (+) Analytical results : not more difficult than static copula
- (+) Have the "fat tail effect"
- (+) Default correlations $\neq 0$ as $t \to 0$ (more stable)
- (-) Only works for small baskets if NtD (FtD ok)
- (-) Increasing "correlation" (ie jump size/intensity) $\Rightarrow$ increases probability of simultaneous defaults
- (+/-) Handling various R_i's requires approximations due to simultaneous defaults

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Disclaimer

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Thank you for your attention