On the Consistency of "European Proxy" of Structural Models for Credit Derivatives

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Abstract

In this paper, we focus on what we call "European Proxy" of structural models, or shortly "Proxy-Structural models" for credit derivatives. In standard structural models, default arrives as the first hit of a stochastic process to a barrier, hence involving a path-dependent condition. In "Proxy-Structural models", by contrast, the *path* condition modeling the default indicator function is replaced by a *point-wise* criterion. This approximation has been considered by some authors to be successful in the sense that proper calibration of those on CDO market is fast, simple and yields meaningful results in terms of implied prices and parameter movements. Some people may not find them intuitive but up to now, there was no reason to question their relevance for credit derivatives as a whole.

In this paper, we show that this class of models exhibit a philosophical problem, which might potentially have an impact when pricing some specific multivariate credit products other than CDO tranches.

Index Terms

Credit Derivatives, Collateralized Debt Obligation, n^{th} to default, Merton models, Dynamic models, Gamma processes.

I. Introduction

In this paper, we focus on what we call "Proxy-Structural models" for credit derivatives. In such a model, the default probability by time t is modeled via a *point-wise* criterion rather than the more standard "first passage time" approach. Their major advantage is to enter the

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framework of a semi-analytical pricing developed for copula models. They have been applied to the valuation of Collateralized Debt Obligation (CDO) tranches, for example.

However, this class of models do suffer from drawbacks. For example, the approach is not really intuitive (there is no "event-based" interpretation anymore, models only define *probabilities*), and consequently, there is no strict Monte Carlo equivalent which could be used for checking the semi-analytical implementation. These drawbacks can be considered as minor issues, as they may not entail the ability of the model to be properly applied to the valuation of products. On the contrary, a more annoying issue would be raised if this class of models could return *inconsistent* results. This type of models could hence be applied to a specific class of structured credit products only, with the obvious problem that this "class of products" is *a priori* not clearly defined. The main contribution of this paper is to effectively show that these models suffer from internal inconsistencies.

The remaining of this paper is organized as follows. First, factor models are recalled in the context of credit derivatives, that is when the goal is to model default times (probabilities). Next, in Section III, "Proxy-Structural" models are defined. Finally, Section IV discusses the application of the Gamma-model in the context of NtD trades pricing. It is shown that the model may return negative conditional probabilities, which could matter in some cases.

II. Factor models for pricing credit derivatives

In order to price a multi-underlying structured credit product, two key steps have to be achieved.

First, univariate default probabilities are obtained by modeling the default event $\{\tau_i \leq t\}$, defined as a function of τ_i, the (random) default time of name i. The probability of this event can be obtained from the Credit Default Swap (CDS) market. A default probability (DP) curve

$$F_i(t) \doteq \Pr[\tau_i \leq t] \tag{1}$$

can be estimated. Typically, this is done by letting τ_i be the first jump of a Poisson process, $X_i(t)$, with piece-wise constant rate $\lambda(t)$ and initial value $X_i(0) = 0$. In that case, the default probability is given by

$$F_i(t) = \Pr[X_i(t) > 0] = 1 - \Pr[X_i(t) = 0] = 1 - \mathrm{e}^{-\int_{s=0}^{t} \lambda(s) ds} \quad . \tag{2}$$

The second step in the pricing of multiple-name credit derivatives (including N underlying obligors) one needs additional information compared to the gathering of all the curves $\{F_1(t), \ldots, F_N(t)\}$. Generally speaking, one needs the N-dimensional multivariate default distribution. To that end, factor models can be used. They can be either static or dynamic.

In static models, the default event is modeled as follows :

$$\{\tau_i \leq t\} = \{Z_i \leq K_i(t)\} \ . \tag{3}$$

with the calibration equation for the barrier functions $K_i(t)$

$$\Pr[Z_i \leq K_i(t)] = F_i(t) \ . \tag{4}$$

The purpose of the Z_i variable is to synthesize the information about the coupling of the default times :

$$Z_i = c_i Z + \tilde{c}_i \tilde{Z}_i \ , \tag{5}$$

where typically $Z, \tilde{Z}_i$ are independent variables, and $c_i, \tilde{c}_i$ are scaling coefficients controlling the variance of Z_i together with the correlation between Z_i and Z_j, resulting from the variable Z which appears in each Z_i, $i \in \{1, \ldots N\}$. When $Z, \tilde{Z}_1, \ldots, \tilde{Z}_N$ are independent standard normal variables, $c_i \in [0, 1[$ and $\tilde{c}_i = \sqrt{1 - c_i^2}$, we have the (at least formerly) popular One-Factor Gaussian Copula model. A lot of variants are possible by modifying the distribution of the latent variables or by increasing their number in each Z_i.

One readily sees that conditional upon Z, defaults are mutually independent. This class of models is referred to as "static", since the distribution of the hidden variables (including Z, which intuitively represents the "state of the economy", because low value of Z tends to increase conditional DP curve, and reversely for higher values of Z) is fixed and do not evolve in time.

Dynamic models, by contrast, suppose all the hidden variables are time-dependent (hence, in fact, are *stochastic processes*). For instance, the dynamic counterpart of eq. 5 is

$$Z_i(t) = c_i Z(t) + \tilde{c}_i \tilde{Z}_i(t) \ , \tag{6}$$

where now $Z(t), \tilde{Z}_1(t), \ldots, \tilde{Z}_N(t)$ are stochastic processes, typically with initial value zero, independent and identically distributed (i.i.d.) increments.

Because of this time dependency, one must adapt the modeling of default dependence. A standard alternative to the static equation eq. 3 is

$$\{\tau_i \leq t\} = \{\exists s \in [0,t] : Z_i(s) > K_i(s)\} \ . \tag{7}$$

and the natural counterpart of eq 4 becomes

$$F_i(t) = \Pr\left[\max_{s\in[0,t]}\Big(Z_i(s) - K_i(s)\Big) > 0\right] \ . \tag{8}$$

This expression is that of a structural model : default time is modeled as the *first passage time* of a process $Z_i(t)$ above a barrier $K_i(t)$. Most of the time, calibration yields that $K_i(t)$ is indeed non-constant with respect to time, except for the hypothetical dynamic model such that, at each time, the rate of default exactly corresponds to the slope of the DP curve at that time.

Typically, these models are solved via Monte Carlo pricing. Paths are drawn from the constitutive equations of the model, and for each trial, one records the minimum of the contract maturity and the first passage time and plug the latter into the cashflow model. The key problem however, is that these models, in order to be realistic, are sufficiently complex such that no closed-form exists for the RHS of eq. 8. They become almost intractable if calibration needs to be done, as it is usually the case.

In order to circumvent this issue, some researchers suggest to modify this model so that a semi-analytical pricing method can be developed, while keeping the dynamic feature of the model. This gave birth to "Proxy-Structural" models. They are defined in the section below.

III. Proxy-Structural models

In order for these attractive models to be used, some researchers suggest to replace the RHS of eq. 8 by a "European approximation" of the form (see for example [Baxter2006], [Baxter2007], [Baxter2008])

$$F_i(t) = \Pr[Z_i(t) > K_i(t)] \ . \tag{9}$$

In this equation, one forgets about *paths*, and focuses on the state of the processes at *instantaneous moment in time*. One may be concerned by the fact that depending on the shapes of the barriers and the dynamics of the stochastic processes involved in the model, one could potentially have, for some $t > 0, \Delta t > 0$

$$Z_i(t) > K_i(t) \text{ , while } Z_i(t+\Delta t) < K_i(t+\Delta t) \ . \tag{10}$$

The indicator function $\mathbb{1}_{\{Z_i(t)>K_i(t)\}}$ may not be an increasing function of time for some generated paths, while obviously, so is $\mathbb{1}_{\{\tau_i \leq t\}}$: the last function always jumps from 0 to 1 at the realized default time $t = \tau_i$, and remains constant for $t > \tau_i$.

On the other hand, after all, this may not be such a big problem. Indeed, the key point in this approach is not to model the *default event* directly, but rather to model its *probabilities*. For instance, the consequence of this is that one cannot transpose this model into a Monte Carlo framework anymore, since there is no clear definition of "default" (for example, because of eq. 10, one could have multiple points at which barrier is reached from below, triggering an ambiguity about the – unique– default time). Nevertheless, in spite of this abnormality, one could still decide to use it, because proper calibration of barrier functions $K_i(t)$ to DP curves $F_i(t)$ will insure that $\Pr[Z_i(t) > K_i(t)]$ will be increasing functions of time in $[0, 1]$. Knowing the cumulative distribution $F_{Z_i(t)}(x)$ of $Z_i(t)$ at each time t and its quantile function $F^{[-1]}_{Z_i(t)}(x)$ (which depends on the parameters of the process $Z_i(t)$), one could now fully enjoy the power of this assumption to compute the barrier functions:

$$K_i(t) = F^{[-1]}_{Z_i(t)}\Big(1 - F_i(t)\Big) \ .$$

This is in a lot of cases very tractable, at least numerically, if not theoretically.

Moreover, one directly sees from eq. 6 that conditional upon $Z(t)$, defaults arrive independently up to t. For instance,

$$\Pr[\tau_i \leq t | Z(t) = z] = 1 - F_{\tilde{Z}_i(t)}\left(\frac{K_i(t) - c_i z}{\tilde{c}_i}\right) \ . \tag{11}$$

and hence the no-default probability up to t is given by :

$$\Pr\left[\min_{i\in\{1,\ldots,N\}} \tau_i > t \Big| Z(t) = z\right] = \prod_{i=1}^{N} F_{\tilde{Z}_i(t)}\left(\frac{K_i(t) - c_i z}{\tilde{c}_i}\right) \ . \tag{12}$$

Next section discusses the relevance of the valuation of NtD via this class of model.

IV. Consistency Analysis via NtD pricing

Although first presented for valuing CDO tranches, it is tempting to use a "Proxy-Structural" model to price some variants, like NtD. In this section, we first recall NtD key equations, before showing that this model is not be appropriate to value this product when various recoveries are involved. Furthermore, it is shown that internal inconsistencies, such as negative probabilities, might be found in this model.

A. *pricing NtD*

Denote by $\pi_1, \dots \pi_N$ the permutation of $\{1, \dots, N\}$ such that $\tau_{\pi_1} \leq \dots \leq \tau_{\pi_N}$. The protection and premium legs of a NtD are, in the case all recoveries are identical ($R_i = R$) :

$$Prot(n) = (1-R) \int_{t=0}^{T} \delta(t) f^{(n)}(t) dt$$
$$Prem(n) = sp \sum_{j=1}^{J} \delta(t_j) \left(\Delta(t_{j-1}, t_j) - \int_{t=t_{j-1}}^{t_j} \mathbb{P}\left[\tau_{\pi_n} \leq t\right] dt \right) \quad .$$

In the above equations, $f^{(n)}(t)$ stands for the density of the nth default time, $\delta(t)$ is the discount factor at t, sp is the contract spread, $\Delta(t, t')$ is the time lag (in years), between t and t', and R_i is the (fixed) recovery rate of name i.

When the recoveries are name-specific, the expected loss can be splitted among names : each individual LGD is weighted with respect to the probabilities for each of them to cause the n-th default :

$$Prot(n) = \int_{t=0}^{T} \delta(t) \left(\sum_{i=1}^{N} (1 - R_i) f_{n,i}(t) \right) dt \quad , \tag{13}$$

where $f_{n,i}(t)$ stands for the density that the i-th name will be the n^{th} one to default by time t. This density can be obtained via our model as

$$\begin{aligned} f_{k,i}(t) &= \lim_{dt \to 0} \frac{\Pr[\tau_i \in [t, t+dt[, \sum_{j \neq i} \mathbb{I}_{\{\tau_j \leq t\}} = k-1]}{dt} \\ &= \int_z f_{i|Z(t)}(t,z) \Pr\left[\sum_{j \neq i} \mathbb{I}_{\{\tau_j \leq t\}} = k-1 \middle| Z(t) = z \right] dF_{Z(t)}(z) \end{aligned} \tag{14}$$

where $f_{i|Z(t)}(t,z)$ is the time-derivative of the conditional default probability :

$$f_{i|Z(t)}(t,z) \doteq \frac{d \Pr[Z_i(s) \leq K_i(s) | Z(t) = z]}{ds} \Big|_{s=t} \quad .$$

If we define $N(t)$ the number of default by time t, we have

$$N(t) = \sum_{i=1}^{N} \mathbb{I}_{\{\tau_i \leq t\}} \ , \quad \mathbb{P}[\tau_{\pi_n} \leq t] = \mathbb{P}[N(t) \geq n] = \sum_{i=n}^{N} \mathbb{P}[N(t) = i] \quad .$$

These probabilities can be obtained similarly than in the CDO case (ie via an ad-hoc recursion algorithm or via FFT, see for instance [Hull and White], [Andersen et al.]).

B. An example : the gamma case

In this model, $c_i = \tilde{c}_i = 1$ and $Z(t), \tilde{Z}_i(t)$ are independent gamma processes, distributed as

$$Z(t) \sim \mathbf{\Gamma}(x; \phi\gamma t, \theta)\ ,\quad \tilde{Z}_i(t) \sim \mathbf{\Gamma}\left(x; (1-\phi)\gamma t, \theta\right)\ ,\quad \theta > 0\ .$$

This kind of processes have been extensively studied; for more details we refer to the monograph [Schoutens]. Theoretical results show that

$$Z_i(t) \sim \mathbf{\Gamma}(x; \gamma t, \theta)\ ,\quad \mathrm{corr}(dZ_i(t), dZ_j(t)) = \phi\ .$$

The conditional probability is given by

$$\Pr[\tau_i \le t | Z(t) = z] = \Pr[Z_i(t) > K_i(t) | Z(t) = z] = 1 - \mathbf{\Gamma}\left(K_i(t) - z; (1-\phi)\gamma t, \theta\right)\ ,$$

with $K_i(t) \doteq \mathbf{\Gamma}^{[-1]}\left(1 - F_i(t), \gamma t, \theta\right)$. We note that the only missing term is the conditional density $f_{i|Z(t)}(t|z)$.

Using the properties of gamma process, we have that $Z_i(t+dt) = Z(t) + \tilde{Z}_i(t) + \Gamma(dt)$ where $\Gamma(dt) \sim \mathbf{\Gamma}(x; \gamma dt, \theta)$, and it comes

$$\begin{aligned}\Pr[\tau_i \in [t+dt] | Z(t) = z] =& \Pr[\tau_i \le t + dt | Z(t) = z] - \Pr[\tau_i \le t | Z(t) = z] \\ =& \mathbf{\Gamma}\left(K_i(t) - z; (1-\phi)\gamma t, \theta\right) - \Pr[\tilde{Z}_i(t) + \Gamma(dt) \le K_i(t+dt) - z] \\ =& \mathbf{\Gamma}\left(K_i(t) - z; (1-\phi)\gamma t, \theta\right) - \mathbf{\Gamma}\left(K_i(t+dt) - z; ((1-\phi)t + dt)\gamma, \theta\right)\end{aligned} \tag{15}$$

implying (see the appendix)

$$f_{i|Z(t)}(t|z) = \frac{\boldsymbol{\gamma}(K_i(t) - z; (1-\phi)\gamma t, \theta)}{\boldsymbol{\gamma}(K_i(t); \gamma t, \theta)}\left(F_i'(t) + \gamma\xi(K_i(t), t, 0, \gamma, \theta)\right) - \gamma\xi(K_i(t) - z, t, \phi, \gamma, \theta) \tag{16}$$

With these results in hand, we are now able to value a NtD trade with the "gamma Proxy-Structural" model. This is done in the next section.

C. *Discussion*

Protection and premium legs are closely linked. In some circumstances, for instance, some quantities can be computed from both legs. If this quantity appears to be different when inferred from premium and protection legs, this would clearly show that (at least) one of the two ways for computing this figure is wrong. More exactly, there would be a clear objection in using the model[1].

In order to rule out any implementation problem related to the recursion algorithm we consider in the following a First-to-Default (FtD, $n = 1$) trade. We further suppose $\delta(t) = 1$ (no interest rate) and homogeneous recoveries ($R \doteq R_1 = \ldots = R_n$). In this case, premium and protection legs can be written as

$$\begin{aligned} Prot(1) &= (1-R)\sum_{i=1}^{N}\int_{t=0}^{T} f_{1,i}(t)dt \\ Prem(1) &= sp\sum_{j=1}^{J}\delta(t_j)\left(\Delta(t_{j-1},t_j) - \int_{t=t_{j-1}}^{t_j}\underbrace{\mathbb{P}\left[N(t)\geq 1\right]}_{(i)}dt\right) \\ f_{1,i}(t) &= \int_z \underbrace{f_{i|Z(t)}(t,z)}_{(iii)}\underbrace{\Pr\left[\sum_{j\neq i}\mathbb{I}_{\{\tau_j\leq t\}}=0\Big|Z(t)=z\right]}_{(ii)}dF_{Z(t)}(z)\ . \end{aligned} \tag{17}$$

In this case, (i) is given by integrating with respect to $Z(t)$ the following conditional distribution :

$$\Pr[N(t)\geq 1|Z(t)=z] = 1-\Pr[N(t)=0|Z(t)=z] = 1-\prod_{i}^{N}\Big(1-\Pr[\tau_i\leq t|Z(t)=z]\Big)\ , \tag{18}$$

while (ii) is given by

$$\Pr\left[\sum_{j\neq i}\mathbb{I}_{\{\tau_j\leq t\}}=0\Big|Z(t)=z\right] = \prod_{j\neq i}^{N}\Big(1-\Pr[\tau_j\leq t|Z(t)=z]\Big)\ . \tag{19}$$

The only remaining challenge is the computation of (iii), namely $f_{i|Z(t)}(t|z)$.

Furthermore, under the specific assumptions related to our test case, the protection leg simply becomes

$$Prot(1) = (1-R)\,\mathbb{E}\left[\mathbb{I}_{\{\tau_{\pi_1}\leq t\}}\right] = (1-R)\left(1-\mathbb{P}[N(t)=0]\right)\ . \tag{20}$$

[1]Careful implementation and checks revealed that numerical errors could not be responsible for this difference.

	Gaussian	Double T	Static Gamma	"Proxy Merton" Gamma
Premium	45,86%	46,01%	42,40%	42,33%
Prot	45,86%	46,00%	42,39%	61,31%

TABLE I

ESTIMATION OF $\Pr[N(T) \geq 1]$ FROM FTD PRICING : $N = 5$, $T = 3y$, $\delta(t) = 1$, QUARTERLY PAYMENT (0.25 YEARS), CONSTANT HAZARD RATES ARE $[1,7\%, 8,9\%, 3,1\%, 9,2\%, 4,5\%]$, NO INTEREST RATES, HOMOGENEOUS 40% RECOVERY. $c_i = 0.6, \tilde{c}_i = \sqrt{1-c_i^2}$ (GAUSSIAN), $c_i = 0.6\sqrt{\frac{\nu-2}{\nu}}, \tilde{c}_i\sqrt{1-0.6^2}\sqrt{\frac{\nu-2}{\nu}}$, $\nu = 4$ (DOUBLE T), $\phi = 0.5$, $\gamma = 2$, $\theta = 1$ (STATIC AND DYNAMIC GAMMA).

A consistency check can be done by checking if the following relation always holds:

$$\int_{z=-\infty}^{\infty} \underbrace{\prod_{i=1}^{N} \left(1 - F_{i|Z(T)}(T,z)\right)}_{\Pr[N(T)=0|Z(T)=z]} dF_{Z(T)}(z) \stackrel{(a)}{=} \Pr[N(T)=0] \stackrel{(b)}{=} 1 - \int_{t=0}^{T} \sum_{i=1}^{N} f_{1,i}(i,t)dt \quad (21)$$

where equality (a) results from conditional independence (eq. 12), and (b) exploits eq. 17 and eq. 20.

Numerical tests show that the two integrals in the above equation do not lead to the same value for $\Pr[N(t)=0]$. Table I shows this discrepancy in the pricing of a 3-year FtD with quarterly payments. Numerical integrations related to unconditioning is computed via a quadrature rule with 64 (Gaussian and Double T copula) or 256 (Gamma-based) points. The used rules are Gauss-Hermite (Gaussian), Mapped-Legendre (Double T, for more details on that technique, we refer to [Vrins]), or Gauss-Laguerre (Gamma-based).

If one agrees that numerical errors may not be responsible for this difference (see below), this result shows that one of the two equalities (a) or (b) is wrong.

In order to figure out which one, we shall exploit the relationship between this dynamic version and its static counterpart. We focus on eq. 21 with $t = 1$. The static gamma model uses constituting equation eq. 5 with $Z \leftarrow Z(1)$ and $\tilde{Z}_i \leftarrow \tilde{Z}_i(1)$ where the processes are defined in Section IV-B. Obviously, eq. 21 must hold under both models.

On the one hand, it can be proved that the integrals related to (a) are the same under both static and dynamic models. Indeed, the constant latent variables involved in the static models match the latent processes involved in the dynamic model evaluated at time $t = 1$ are the same, and hence so are the related unconditional and conditional default distributions in eq. 18. Numerical experiments confirm this statement. Obviously, by transitivity, one expects the integral of eq. 21 related to equality (b) to yield the same result under both models as well.

On the other hand, by contrast, this is intuitively very unlikely. In the dynamic case indeed, the computation of the conditional density at t requires conditioning "at each time t", and this involves time-varying distributions $F_{Z(t)}(x)$ while in the static case, the distribution of the conditioning variable is that of $Z(1)$ for all t. By contrast, in the evaluation of the left-most integral a "unique conditioning at maturity" was needed : the unconditional distribution of no default *up to* $T = 1$ was computed using conditional distributions with respect to a variable with the distribution of $Z(1)$, under both models. This development intuitively shows that there must exist an ambiguity in the evaluation of $\Pr[N(1) = 0]$, or more precisely that the contingent leg is not consistent with $\Pr[N(1) = 0]$ as computed from the recursion algorithm. Numerical experiments confirm this view.

Summarizing, integrals related to (a) are the same under both static and dynamic models, but not those related to (b). A formal proof of this claim is rather difficult to achieve; it would require to show that

$$\int_{t=0}^{1} \sum_{i=1}^{N} \int_z f_{i|Z(t)}(t,z) \prod_{j \neq i} \mathbf{\Gamma} \left(\mathbf{\Gamma}^{[-1]}(1 - F_i(t); \gamma t, \theta) - z; (1-\phi)\gamma t, \theta \right) dF_{Z(t)}(z) dt$$

may not be equal to

$$\int_{t=0}^{1} \sum_{i=1}^{N} \int_z f_{i|Z}(t,z) \prod_{j \neq i} \mathbf{\Gamma} \left(\mathbf{\Gamma}^{[-1]}(1 - F_i(t); \gamma, \theta) - z; (1-\phi)\gamma, \theta \right) dF_Z(z) dt$$

where $Z \sim Z(1)$ and $f_{i|Z}(t|z)$ stands for the conditional density under the static model :

$$f_{i|Z}(t|z) = \frac{\boldsymbol{\gamma}(K_i(t) - z; (1-\phi)\gamma, \theta)}{\boldsymbol{\gamma}(K_i(t); \gamma, \theta)} F_i'(t) \quad , \tag{22}$$

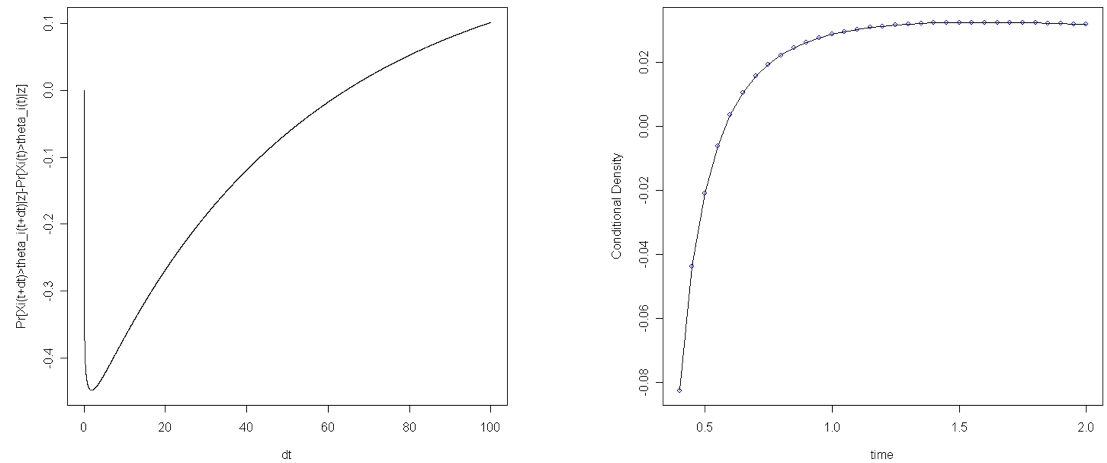
which is the static counterpart of eq. 16.

More generally, in a homogeneous recovery scheme and with no interest rate, $\Pr[N(T) = 0]$ as given by the recursion algorithm in the dynamic gamma case (with shape parameter of the driving process set to γ) matches that of the corresponding static gamma model with shape parameter set to $\gamma \leftarrow \gamma T$ (set $Z \leftarrow Z(T)$, $\tilde{Z}_i \leftarrow \tilde{Z}_i(T)$). This probability is also in line with that obtained via the protection leg under the static model. By contrast, the result obtained under the dynamic model via the protection leg is different, and here again disagrees with the previously computed figures.

As shown above, numerical tests and intuitive explanation can be used to emphasize the existence of a problem in this model. Actually, a part of the problem is coming from the fact that simultaneous defaults are possible in this model. Therefore, focusing ion FtDs, splitting the first loss among names could not be achieved by summing the probabilities for each of them causing the $1st$ default because of doublons. In other words, one cannot use eq. 13 in this model, and the framework is restricted to pools with same recoveries; the protection leg has to be valued from the first-default time density $f^{(1)}(t)$, available by differentiating the cumulative no-default distribution got from the recursion. Although this is a limitation of the model, it might not be an evidence of a fundamental problem : it could be that the simultaneous defaults are solely responsible for the discrepancies, which could happen in any other models in which defaults are resulting from a common systemic jump process. By contrast, there is still a problem in there. This can be seen by looking at the behavior of the conditional probability as a function of time, or more precisely at the expression $\Pr[\tau_i \in [t, t+\Delta t]|Z(t) = z]$, as given in eq. 15. Obviously, one expects this expression to be increasing with Δt. Surprisingly, it can be shown that there exists some states $Z(t) = z$, some time t, well-behaved single-name CDS default probability curves $F_i(t)$ (to which $K_i(t)$ barriers are associated) and parameters γ, ϕ, θ for which this probability is non-increasing with respect to Δt. This is illustrated in Figure 1(a).

Remark : impact of numerical errors

One could potentially question the accuracy of the numerical estimation of the conditional density $f_{i|Z(t)}(t, z)$ of the dynamic gamma model (see eq. 16). Throughout this paper, numerical figures have been produced thanks to the reliable and widely used statistical software *R* [R-Project]. If one trusts the numerical results returned for the built-in gamma distributions and densities, the potential issues are confined in the computation of the function $\xi(\ldots)$, which is involved in the expression of the conditional density (see the appendix). The evaluation of this function requires a numerical integration of the product of a gamma density and the logarithm, between 0 and some upper integration bound. Nevertheless, one can rule out potential effects due to implementation either by using an adaptive quadrature (for which the error can be controlled) or



(a) Conditional default probability curve is not always increasing

(b) Conditional Density can be negative

Fig. 1. Left : Probability $\Pr[\tau_i \in [t, t+\Delta t]|Z(t) = z]$ defined as $\Pr[X_i(t+\Delta t) > K_i(t+\Delta t)|Z(t) = z] - \Pr[X_i(t) > K_i(t)|Z(t) = z]$ vs Δt. This probability is not necessarily increasing with Δt in the case $t = 0.175$. Right : Conditional density $f_{i,Z(t)}(t, z)$ vs time ; this density is not positive for all t (dots : empirical with $dt \leftarrow 1E^{-10}$ (ratio of RHS of eq. 15 by dt), solid curve : theoretical, as given by eq. 16). Parameters are defined as follows : $\phi = 0.2, \gamma = 2, \theta = 1$, $z = 3.87$ and the constant hazard rate is set to $\lambda = 0.017$

ii) by using the built-in R function *integrate()*, which provides the user with an upper bound for the absolute error made. With those tools, one can see that in our context, the numerical errors could not be held responsible for the negative conditional density drawn in Fig. 1(b).

V. Conclusion

This paper deals with "Proxy-Structural models". This naming refers to modeling schemes in which probabilities that an entity defaults by time t is set equal to that of a non-trivial stochastic process (i.e. increments are not forced to be 0) to be larger than a threshold at that time t, hence discarding the path of this process up to t.

This class of models have been proven to give interesting results in terms of CDO tranche fitting. They allow good market calibration while they exhibit a reasonable parameter movements accross the recent market turmoil. Therefore, it is tempting to apply this class of model to some variants, like NtD.

Nevertheless, the basic assumption behind this kind of models is a bit counter-intuitive, and there is no guarantee that proper calibration to CDS curves prevent inconsistent results on some particular credit derivatives basket, similar than (but different to) CDO tranches. It has been emphasized that such kind of models may not be appropriate for some products as some probabilities could become negative. Our interpretation of this issue is that by modeling only specific (here, unconditional) *probabilities of events* instead of *events* themselves, one could not guarantee that all derivative (e.g. conditional) probabilities will be meaningful.

APPENDIX : CONDITIONAL DENSITY OF THE DYNAMIC GAMMA MODEL

We first note that in the hazard rate model for single-name default probabilities, we have

$$F_i(t) = 1 - \mathrm{e}^{-\int_{s=0}^{t} \lambda(s) ds} \quad ,$$

and therefore

$$F_i'(t) \doteq \frac{dF_i(t)}{dt} = \lambda(t)\, \mathrm{e}^{-\int_{s=0}^{t} \lambda(s) ds} \quad .$$

The Leibniz rule will be a useful relation hereafter :

$$\frac{d}{dt} \int_{u=a(t)}^{b(t)} f(x,t) dx = \int_{u=a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f(b(t),t) b'(t) - f(a(t),t) a'(t) \quad .$$

Also, if $F^{[-1]}(F(x)) = x$ and $y = F(x)$, then

$$\left.\frac{dF^{[-1]}(y)}{dt}\right|_{y=\tilde{y}} = \frac{1}{\left.\frac{dF(x)}{dx}\right|_{x=F^{[-1]}(\tilde{y})}} = \frac{1}{F'(F^{[-1]}(\tilde{y}))}$$

We are interested in the limit

$$f_{i|Z}(t|z) = \lim_{dt \to 0} \frac{\mathbf{\Gamma}(K_i(t) - z; (1-\phi)\gamma t, \theta) - \mathbf{\Gamma}(K_i(t+dt) - z; (1-\phi)\gamma t + \gamma dt, \theta)}{dt}$$

where the barrier $K_i(t)$ is given by $\mathbf{\Gamma}^{[-1]}(1 - F_i(t); \gamma t, \theta)$. The numerator of limit can be expressed as

$$\begin{aligned} &\mathbf{\Gamma}(K_i(t) - z; (1-\phi)\gamma t, \theta) - \mathbf{\Gamma}(K_i(t+dt) - z; (1-\phi)\gamma t + \gamma dt, \theta) \\ &+ \Bigg(\underbrace{\mathbf{\Gamma}(K_i(t+dt) - z; (1-\phi)\gamma(t+dt), \theta) - \mathbf{\Gamma}(K_i(t+dt) - z; (1-\phi)\gamma(t+dt), \theta)}_{=0} \Bigg) \end{aligned} \quad (23)$$

This limit reduces thus to a sum of two derivatives:

$$f_{i|Z}(t|z) = -\underbrace{\frac{d\,\mathbf{\Gamma}(K_i(t)-z;(1-\phi)\gamma t,\theta)}{dt}}_{\doteq D_1} - \phi\gamma \underbrace{\frac{d\,\mathbf{\Gamma}(K_i(t)-z;k,\theta)}{dk}\Big|_{k=(1-\phi)\gamma t}}_{\doteq D_2} \tag{24}$$

Let us first focus on the computation of the first derivative, D_1 :

$$\frac{d\,\mathbf{\Gamma}(K_i(t)-z;(1-\phi)\gamma t,\theta)}{dt} = \frac{d}{dt}\int_{u=0}^{K_i(t)-z} \boldsymbol{\gamma}(u;(1-\phi)\gamma t,\theta)du \tag{25}$$

By the Leibniz rule, this derivative is equal to

$$\underbrace{\int_{u=0}^{K_i(t)-z} \frac{d}{dt}\boldsymbol{\gamma}(u;(1-\phi)\gamma t,\theta)du}_{\doteq D_{1A}} + \boldsymbol{\gamma}(K_i(t)-z;(1-\phi)\gamma t,\theta)\underbrace{\frac{d(K_i(t)-z)}{dt}}_{\doteq D_{1B}} \tag{26}$$

The integrand of D_{1A} is

$$(1-\phi)\gamma \times \boldsymbol{\gamma}(u;(1-\phi)\gamma t,\theta) \times (\ln(u) - (\ln(\theta) + \psi((1-\phi)\gamma t))$$

where $\psi(x)$ is the *digamma* function : $\psi(x) \doteq \Gamma'(x)/\Gamma(x)$, so that the integral D_{1A} equals

$$(1-\phi)\gamma \underbrace{\left(\int_{u=0}^{K_i(t)-z} \boldsymbol{\gamma}(u;(1-\phi)\gamma t,\theta)\ln(u)du - (\ln(\theta)+\psi((1-\phi)\gamma t))\,\mathbf{\Gamma}(K_i(t)-z;(1-\phi)\gamma t,\theta)\right)}_{\doteq \xi(K_i(t)-z,t,\phi,\gamma,\theta)}$$

Regarding the second term of the RHS of eq. 26, D_{1B}, it is the time-derivative of the barrier $K_i(t)$. In order to compute it, let us define

$$G(x,y) \doteq \mathbf{\Gamma}(x;y,\theta)\ , \quad H(x,y) \doteq \mathbf{\Gamma}^{[1]}(x;y,\theta)$$

We are interested in the derivative of $K_i(t) = H(g(t),\gamma t)$ with respect to time, with $g(t) \doteq 1 - F_i(t)$. Observe that $G(H(g(t),\gamma t),\gamma t) = g(t)$.

We have

$$\begin{aligned}\frac{dK_i(t)}{dt} &= \frac{\partial H(x,y)}{\partial x}\Big|_{(x,y)=(h(t),\gamma t)}\frac{\partial h(t)}{\partial t} + \frac{\partial H(x,y)}{\partial y}\Big|_{(x,y)=(h(t),\gamma t)}\frac{\partial(\gamma t)}{\partial t}\\ &= -\frac{\partial H(x,y)}{\partial x}\Big|_{(x,y)=(h(t),\gamma t)}F_i'(t) + \boldsymbol{\gamma}\,\frac{\partial H(x,y)}{\partial y}\Big|_{(x,y)=(h(t),\gamma t)}\end{aligned}$$

Using the rule for the derivative of inverse functions, we have

$$H_x \doteq \frac{\partial H(x,\gamma t)}{\partial x} = \frac{1}{G_x(H(x,\gamma t),\gamma t)}$$

Furthermore, we have

$$G_x(H(x,\gamma t),\gamma t)H_y(x,\gamma t)\gamma + G_y(H(x,\gamma t),\gamma t) = 0$$

But, it can be shown that

$$G_y(x,y) = \int_{u=0}^{x} \gamma(u;y,\theta)\ln(u)du - (\ln(\theta)+\psi(y))\,\mathbf{\Gamma}(x;y,\theta)$$

Hence,

$$\frac{dK_i(t)}{dt} = \frac{dH(1-F_i(t),\gamma t)}{dt} = \frac{-F_i'(t) - \gamma G_y(K_i(t),\gamma t)}{\gamma(K_i(t);\gamma t,\theta)} = \frac{-F_i'(t) - \gamma\xi(K_i(t),t,0,\gamma,\theta)}{\gamma(K_i(t);\gamma t,\theta)} \tag{27}$$

This concludes the computation of the derivative D_1 as appearing in eq. 24. The second derivative, D_2 simply reduces to

$$\frac{d\,\mathbf{\Gamma}(K_i(t)-z;k,\theta)}{dk}\Big|_{k=(1-\phi)\gamma t} = \xi(K_i(t)-z,t,\phi,\gamma,\theta)\ , \tag{28}$$

which yields the density $f_{i|Z}(t|z)$ in eq. 24 is given in eq. 16.

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