# Understanding decreasing CDS curves

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**Abstract**

One of the major input for evaluating a Credit Default Swap (CDS) position is the so-called CDS curve. This curve gives the term structure of the CDS: for some maturities (typically: 1 year, 2 years, 3 years, 4 years, 5 years, 7 years and 10 years) a market spread is given. The spread is the premium to pay to a counterparty to protect one unit of currency during one year.

Most of the time, CDS curves are increasing: the larger the maturity (i.e. the longer the time period we want to protect against a credit event), the larger the spread. As from the beginning of the Credit crisis (Summer 2007), some CDS curves were reverted, meaning that they contained decreasing parts: the spread (premium given on an annual basis) to pay for having protection for $m_1$ years is larger than the spread to pay for having protection for $m_2 > m_1$ years. In some pathological cases, when the reversion is quite important, these reverted curves caused the blocking of some CDS pricers. The goal of this report is to understand these pricer blockings from a practical and quantitative perspective.

Our analysis is based on a market-driven model for pricing CDS products, assuming a piecewise linear cumulative density function for the implied default probabilities. In spite of its simplicity, the model is quite general because only few assumptions are made, and provides manageable closed-form expressions for some interesting quantities. In this paper, the following key results will be derived:

- Closed-form solutions are obtained for some important quantities, allowing a deep understanding of the effects of the underlying parameters;
- It is shown that there is no theoretical objection to the existence of decreasing parts in CDS curves;
- However, indeed, there exists a threshold on the intensity of these reversions : the CDS curves cannot contain parts being arbitrarily decreasing. If the CDS curve contains too strongly reverted parts than allowed (according to the above-mentioned threshold), then it is natural that the pricer fails to return a valid cumulative default probability curve;
- Finally, a business meaning of this "reverting threshold" is given in terms of arbitrage opportunities.

Note: This paper suits to anyone being interested in credit derivatives product, and does not require any specific prerequise.

## 1 Introduction

A Credit Default Swap (CDS) on a company $n$ is a financial product in which a protection buyer agrees to pay as from the effective date $t_0$ till the maturity date $t^\star$ periodic premiums at some standardized coupon dates $t_1, \ldots, t_N$ falling between these two dates to a protection seller (with $t_N = t^\star$). The company $n$ is

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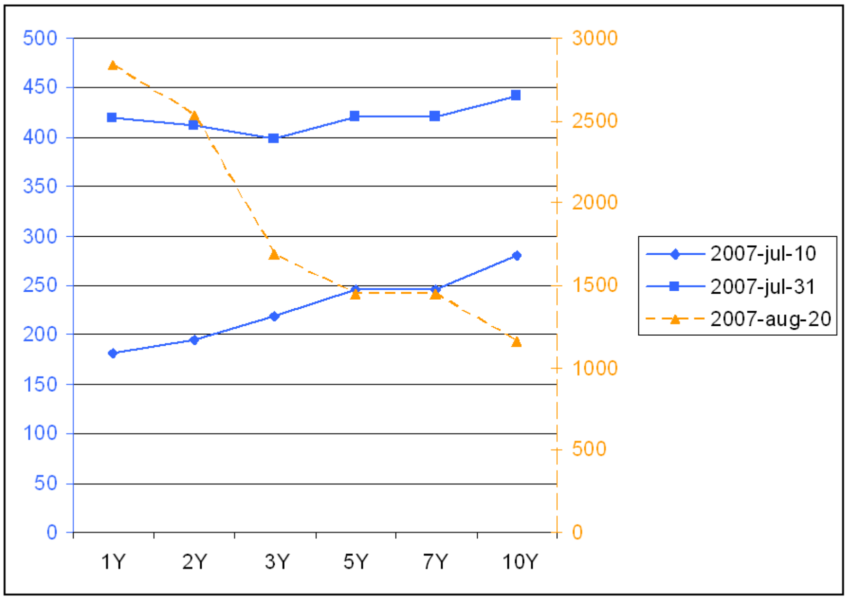



Figure 1: CDS curves for Residential Capital LLC for quoted dates 2007/07/10, 2007/07/31 and 2007/08/20. The spread values are given in bps ($Y$-axis) for each available maturity ($X$-axis). The dashed curve is linked to the right-hand side axis. The curves are obtained through linear interpolation between market quotes (labeled with markers).

assumed to not default before $t_0$, otherwise, the transaction is canceled. If $n$ defaults at $t_0 \leqslant \tau \leqslant t^\star$, then only the premiums for the coupon dates coming before $\tau$ will be paid by the protection buyer, and a final payment is made to cover the protection offered between $\tau$ and the previous coupon date. If $n$ defaults between $t_0$ and $t^\star$, the protection seller agrees to pay an amount covering the losses of $n$ to the protection buyer at the next coupon date after $\tau$. If $n$ does not default before $t^\star$, then all the premiums are due. On the other hand, in that case, the protection seller does not have to pay anything. In the above transaction, we assume that the protection also covers the days $t_0$ and $t^\star$. The net present value (NPV) of the payments made by the protection buyer is called the Fee leg, while the total value of the payment made by the protection seller is the Contingent leg. These legs depend, among other, on the protection duration. The periodic payments are proportional to a spread, which is given for a specific maturity. For each CDS, several spreads are quoted: one for each of the available maturities. The curve built by plotting the spreads with respect to their associated maturity is called the *CDS curve*. Figure 1 gives such CDS curves related to the company *Residential Capital LLC* quoted on three different dates. On one given curve, the value associated to each marker is the market spreads (in bps) for the corresponding protection duration. The curves show the linear interpolation between these quoted values. For more details about CDS, we refer to the handbooks [Francis C. et al. (2003)], [Masters B. et al. (2000)] and the books [Bieleki T.R. and Rutkowski M. (2005)], [Choudhry M. (2006)].

Since the last few months some CDS spread tend to deviate from their average level (Fig . 1 shows that the average spread level of *Residential Capital LLC* widened by a factor 8 between July 10th and August 20th at the beginning of the 2007 Summer Crisis). Very high spreads have been observed for some names involved in CDS deals. But more surprisingly than this observed "upper shift" of the CDS curve, the *shape* of these curves became unusual. Indeed, in the past, they used to have an increasing trend : the longer the protection period, the larger the spread (see e.g. the CDS curve labeled with "$\diamond$" markers on Fig. 1). This smoothly increasing trend was intuitively understood as the increasing price to pay for a longer protection period. Consequently, some analysts were surprised to recently observe CDS curves containing decreasing parts (see e.g. the CDS curve labeled with "□" or "△" markers on Fig. 1). Also, quantitative analysts came into troubles when their CDS pricers started to fail to return valid default probabilities. Apparently, most of the CDS pricers blocked on CDS curves that included strongly decreasing parts.

The goal of this report is to understand these pricer blockings from a practical and quantitative perspective. To that end, we propose a three-fold analysis. First, in Section 2 a practical model for pricing CDS products is proposed. It includes common market conventions: for example, cash flows can only happen at specific coupon dates, a discrete-time world (day to day) is assumed. The model assumes a piecewise linear cumulative default probability function. In spite of its linearity, however, the model is quite general, and leads to results being pretty well in line with more involved models (some examples of alternative models can be found in [Hull J.C. and White A. (2003)], [Hull J.C. et al. (2004)] [Cariboni (2004)], [Bieleki T.R. et al. (2005)], [Bieleki T.R. and Rutkowski M. (2005)] and references therein). We do believe that this is a consequence of that few assumptions are needed to derive it. Another advantage of focusing on such a simple model is that closed-form expressions can be found. In the second part of the paper (Section 3), we investigate, from both qualitative and quantitative points of view, the possibility for a CDS curve to have decreasing parts. Finally, an analytical threshold is given for the CDS curve reversion, and the pricer blockings are explained according to this value. Further, a business meaning is proposed for this threshold (and consequently for the pricer blockings) in terms of arbitrage opportunities (Section 4).

The following "contract conventions" are made in our developments:

$\mathcal{C}$onv1 The time scale is discrete, not continuous. This is, in practice, a very natural practical convention: we need to pick up a precise moment for the beginning and the end of the transactions, and to act a possible credit event. This is not possible to do, in practice, on a continuous scale. Therefore, the counterparties have to agree on the (maximum) resolution of this time scale. Evolving in time by a day-to-day scale is certainly the most common convention (in case of credit event, for instance, only the default date will matter: any further time information about when the default would have occurred during that day will be discarded). Consequently, we will also adopt this time resolution, so that only dates matters in computing the cashflows[1];

$\mathcal{C}$onv2 cash flows can only be made on specific (agreed beforehand) coupon dates;

$\mathcal{C}$onv3 we assume a constant number of days in a year (noted $\gamma$);

Similarly, the pricing model assumes the following:

$\mathcal{A}$ss1 the bid/ask spreads are the same;

$\mathcal{A}$ss2 the cumulative default probability curve is linear between each of the possible maturity dates (piecewise linear). This is nothing but assuming a parametric family for this curve;

$\mathcal{A}$ss3 the market is efficient : the spreads reflects the ask-and-demand law (hence, liquidity is assumed), there is no arbitrage opportunities "in expected value", ...

# 2 A simple model to infer cumulative default probabilities from CDS curves

Assume we are on date $t_0$ and that we observe a CDS on a name $n$ with spreads available for the $m$ following maturities: $y_1$-year, $y_2$-years, ..., $y_m$-years where $y_i$ stands for a natural number. Practically, the above

[1] This assumption does not entail the generality of the approach, which can easily be extended to a further time scale (hours, minutes,...). However, days are used in the following as the chosen time unit.

contracts deal with protection against a credit event of $n$ till, respectively $t_1^\star = t_0 + y_1$ years, $t_2^\star = t_0 + y_2$ years, ...

Typical values are $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, $y_4 = 5$ and $y_5 = 10$ (leading to $m = 5$). At a given time, say $t_0$, a set of spreads is available for the name $n$:

$$\mathrm{sp}_1(n, t_0),\ \mathrm{sp}_2(n, t_0),\ \mathrm{sp}_3(n, t_0),\ \mathrm{sp}_4(n, t_0) \text{ and } \mathrm{sp}_5(n, t_0)\ .$$

The spreads are given on an annual basis under a given convention about the number of days in a year, noted $\gamma$ (e.g. $\gamma = 360$).

For clarity, the computation of the cumulative default probability function for a given name $n$ will be splitted in two steps. We start by focusing on the first maturity for which the spread is available (say $y_1$). We then consider the remaining maturities by iterating on the results obtained for the previous maturities.

### 2.1 First available maturity

The goal hereafter is to find the (cumulative) default probability at some future date $t_i$ for the first maturity $y_1$ implied by the market. More precisely, we assume that the protection on $n$ for that contract starts at $t_0$ and ends at $t_1^\star$ ($t_0$ included, $t_1^\star$ excluded). In this section, we will answer the following question: how to find today, on date $t_0$, the probability that name $n$ will default between $t_0$ and $t_1^\star$, noted $p_1(n, [t_0, t_1^\star[)$ ? We will proceed step wise, by computing the probability that a name will default between two dates $t_{i-1}$ and $t_i$. These probabilities will be noted $p_1(n, [t_{i-1}, t_i[)$.[2] Because of assumption $\mathcal{A}$ss3, the spread quote $\mathrm{sp}_1(n, t_0)$ is such that there must be no a priori looser or winner in the transaction. Mathematically, this implies that the NPV of the expected Fee leg must equal the NPV of the expected Contingent leg: the two counterparties are expecting a zero Mark-To-Market (MtM) if they would have to enter a CDS contract according to these spreads today, at $t_0$. Hence, $p_1(n, [t_0, t_i[)$ is seen as a tuning parameter for adjusting the MtM to zero; its value is set according to the following formula:

$$p_1(n, [t_0, t_i[) = p \text{ s.t. } \mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n, t_0, t_i, \ldots), \ldots]\} = \mathcal{N}\{\mathbb{E}[\mathrm{CgtLeg}(n, t_0, t_i, \ldots), \ldots]\}\ ,$$

in which $\mathbb{E}[\boldsymbol{\mathcal{E}}]$ denotes the expectation of the stochastic expression $\boldsymbol{\mathcal{E}}$ (which depends on the probability parameter $p$, see below), $\mathcal{N}\{C\}$ stands for the net present value of the cash flow $C$, and $\mathrm{FeeLeg}(n, t_0, t_i, \ldots)$ and $\mathrm{CgtLeg}(n, t_0, t_i, \ldots)$ are the Fee and Contingent legs, respectively, that both depend on some parameters.

Let us now focus on the computation of $p_1(n, [t_0, t_1^\star[)$. In the following two subsections, we shall derive an expression (corresponding to our model for the distribution function of the default probabilities) for the expected Contingent and expected Fee legs.

#### 2.1.1 Computation of the Contingent Leg

Let us define $\boldsymbol{\tau} \geqslant t_0$ the random variable describing the date on which $n$ will default. Then,

$$\mathbb{E}[\mathrm{CgtLeg}(n, t_0, t_1^\star, \ldots), \ldots] = \underbrace{0 \times \mathbb{P}(\boldsymbol{\tau} > t_1^\star)}_{\text{if no default}} + \underbrace{\mathrm{not}(n)(1 - \mathrm{rec}(n)) \times \mathbb{P}(\boldsymbol{\tau} \leqslant t_1^\star))}_{\text{if default}}\ ,$$

where $\mathrm{rec}(n)$ is the recovered amount when $n$ defaults, and consequently, $\mathrm{not}(n)(1 - \mathrm{rec}(n))$ are the protected losses given default.

[2] the bracket on the right of "$[t_{i-1}, t_i[$" means that $t_i$ is not included. This is to avoid to count several times the effect of a default at $t_i$ when summing these probabilities. This convention, combined to the fact that protection ends right before $t_i^\star$ will allow us to not particularize this case in the summations, and consequently, to simplify the notations. As an example, $p_1(n, [t_0, t_1^\star[) = \sum_i p_1(n, [t_{i-1}, t_i[)$ where the sum ranges over the dates $t_{i-1}, t_i$ such that $t_0 \leqslant t_{i-1} < t_i \leqslant t_1^\star$

The NPV of this term can be rewritten as

$$\begin{aligned}\mathcal{N}\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_1^\star,\ldots),\ldots]\} &= \mathcal{N}\left\{\mathrm{not}(n)(1-\mathrm{rec}(n))\sum_{t_0\leqslant t_{i-1}<t_i<t_1^\star}\mathbb{P}(\boldsymbol{\tau}=t_i)\right\}\\ &= \mathrm{not}(n)(1-\mathrm{rec}(n))\sum_{t_0\leqslant t_i\leqslant t_1^\star}p_1(n,[t_{i-1},t_i[)\delta(\lceil t_i\rceil)\ , \end{aligned}\qquad (1)$$

where $t_i$ and $t_{i-1}$ mean any consecutive days between the effective and maturity date, $\lceil t_i\rceil$ is the next coupon date after $t_i$ and $\delta(t)$ is the discount factor that will be applied if a default occurs on $t$, quoted at $t_0$.

According to the convention $\mathcal{C}$onv2, the Contingent leg will be paid only if $n$ defaults before $t_1^\star$, and the corresponding cash flow will occur on the first coupon date after the date of the default. The set of coupon dates for the first maturity of the CDS is noted $\mathcal{T}(1)$ (it includes the dates $t_0$ and $t_1^\star$), where the argument of $\mathcal{T}(\cdot)$ refers to the fact that we are dealing with the first available spread $\mathrm{sp}_1(n,t_0)$ (i.e. the first maturity date).

At this step however, the problem seems yet more complicated than the original one: instead of having to estimate one default probability, we are led to estimate $\sharp[\mathcal{T}(1)]-1$ default probabilities, where $\sharp[\mathcal{S}]$ denotes the number of elements in the set $\mathcal{S}$. Here plays the linear assumption for default probability. We parametrize each of the above probabilities $p_1(n,[t_{i-1},t_i[)$ as a linear function of a single default probability $p$ such that

$$\boxed{p_1(n,[t_{i-1},t_i[)\doteq p\frac{t_i-t_{i-1}}{t_1^\star-t_0}\ ,}\qquad (2)$$

and, by definition, $p=p_1(n,[t_0,t_1^\star[)$ is the probability that $n$ defaults before $t_1^\star$ knowing that it survives till $t_0$. Mathematically, this assumption corresponds to a uniformity of the default occurrences between $t_0$ and $t_1^\star$. As an illustration, whatever is $i$, $p_1(n,[t_{i-1},t_i[)$ remains the same provided that the time gap (in days) between $t_{i-1}$ and $t_i$ does not depend on $i$. For example, the probability to observe a default in $[t_0,t_1^\star/2[$ is half the probability to observe a default in $[t_0,t_1^\star[$.

For simplifying the notation, let us define the shorthand notation "$\sum_{\mathcal{T}(1)}$" for meaning that the summation is performed on each coupon date, that is for "$\sum_{t_0\leqslant t_{i-1}<t_i\leqslant t_1^\star}$" where $\{t_{i-1},t_i\}\subseteq\mathcal{T}(1)$. Hence, with eq. (2) in mind and because of $\mathcal{C}$onv2, the NPV of the expected Contingent leg can be expressed as a sum over coupon dates only, and reduces to[3]

$$\boxed{\mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_1^\star,\ldots),\ldots]\right\}=p\,\mathrm{not}(n)(1-\mathrm{rec}(n))\sum_{\mathcal{T}(1)}\frac{t_i-t_{i-1}}{t_1^\star-t_0}\delta(t_i)\ .}\qquad (3)$$

### 2.1.2 Computation of the Fee Leg

The spreads are assumed to be paid on an annual basis. Hence, the amount to be paid at a coupon date $t_i\in\mathcal{T}(.)$ is (in case of no default) $\frac{t_i-t_{i-1}}{\gamma}\,\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)$ where $\frac{t_i-t_{i-1}}{\gamma}$ is the number of years (according to our convention) between the two consecutive coupon dates $t_{i-1}$ and $t_i$. The expected cash flow for the Fee leg, at each coupon dates, is (see the Appendix for more detailed calculations)

$$0\times\mathbb{P}(\boldsymbol{\tau}<t_{i-1})+\frac{\Delta_i}{\gamma}\,\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\,\mathbb{P}(t_{i-1}\leqslant\boldsymbol{\tau}<t_i)+\frac{t_i-t_{i-1}}{\gamma}\,\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\,\mathbb{P}(\boldsymbol{\tau}\geqslant t_i)\ ,\qquad (4)$$

[3]In this equation, $t_i$ and $t_{i-1}$ denote consecutive coupon dates in $\mathcal{T}(1)$, instead of any consecutive days between $t_0$ and $t_1^\star$, as it was the case in equation 2.

where $\Delta_i$ is defined as the expected number of days between $t_{i-1}$ and the default date when the default occurs in the time interval $[t_{i-1}, t_i[$:

$$\Delta_i \doteq \mathbb{E}[\boldsymbol{\tau} - t_{i-1} | t_{i-1} \leqslant \boldsymbol{\tau} < t_i]$$

and satisfies $0 \leqslant \Delta_i \leqslant t_i - t_{i-1}$. In equation (4), each term denotes the expected amount to pay for the protection between $t_{i-1}$ and $t_i$ depending on when the possible credit event will occur : either the default occurs before $t_{i-1}$ (first term), or during that period (second term) or, finally, after $t_i$ (third term). By playing with $\Delta_i$, one actually tunes the cumulative probability function: by tuning it between 0 to $t_i - t_{i-1}$, the cumulative probability function between those dates is made non-linear. Obviously, this parameter is also inferred by our "piecewise linear" model (see Appendix 7.2). The advantage of this formulation is that the protection buyer can have best and worst cases when the name defaults in that interval by setting $\Delta_i \leftarrow 0$ and $\Delta_i \leftarrow (t_i - t_{i-1})$, respectively (the reverse holds true for the protection seller).

A quite good approximation of $\Delta_i$ according to our model is $\Delta_i = (t_i - t_{i-1})/2$, meaning that if a default occurs between $t_{i-1}$ and $t_i$, it will happen in the middle of two coupon dates. This is because i) we assume a uniform distribution for credit event between the coupon dates ($\mathcal{A}$ss2), and ii) only discount factor $\delta(t_i)$ is involved in the payments related to that time interval ($\mathcal{C}$onv2). The expected payments is simply the sum of the above probabilized amounts which, using our notations for the default probabilities is:

$$\mathrm{sp}_1(n, t_0)\,\mathrm{not}(n)\delta(t_i)\frac{\Delta_i}{\gamma}\times(p_1(n,[t_0,t_i[) - p_1(n,[t_0,t_{i-1}[)) + \mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\times(1 - p_1(n,[t_0,t_i[)) \ .$$

Using our linear model for the probabilities given in eq. (2), the above expression can be simplified as

$$\mathrm{sp}_1(n, t_0)\,\mathrm{not}(n)\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - p\frac{t_i - t_0 - \Delta_i}{t_1^\star - t_0}\right) \ .$$

This term is the discounted expected value of the Fee leg payment regarding the period between the $t_{i-1}$ and $t_i$ coupon dates[4].

By summing over the periods between the consecutive coupon dates involved in the contract, one gets our NPV of the Fee leg payment:

$$\boxed{\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n, t_0, t_1^\star, \ldots), \ldots]\right\} = \mathrm{sp}_1(n, t_0)\,\mathrm{not}(n)\sum_{\mathcal{T}(1)}\left[\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - p\frac{t_i - t_0 - \Delta_i}{t_1^\star - t_0}\right)\right] \ .} \tag{5}$$

### 2.1.3 Computation of the Default probability

By equating the NPV of the above expected Fee and Contingent legs (which is equivalent to find the value of $p$ yielding a zero MtM at time $t_0$), one gets that the corresponding value of $p$ is

$$\boxed{p_1^\star = \frac{\mathrm{sp}_1(n,t_0)\sum_{\mathcal{T}(1)}\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}}{\sum_{\mathcal{T}(1)}\left[(1 - \mathrm{rec}(n))\frac{t_i - t_{i-1}}{t_1^\star - t_0}\delta(t_i) + \mathrm{sp}_1(n,t_0)\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(\frac{t_i - t_0 - \Delta_i}{t_1^\star - t_0}\right)\right]}} \tag{6}$$

and we conclude that

$$p_1(n, [t_0, t_1^\star]) \leftarrow p_1^\star$$

## 2.2 Remaining maturities

Assume we are dealing with the $k$-th maturity ($k > 1$), meaning that we have already found $p_1^\star, \ldots, p_{k-1}^\star$, the cumulative default probabilities for maturities $y_1, \ldots, y_{k-1}$ with maturity dates $t_1^\star, \ldots, t_{k-1}^\star$. Let us compute the Contingent and the Fee legs exploiting this information.

[4] Similar developments can be used to find the expression of this term for the remaining maturities ($k > 1$).

### 2.2.1 Computation of the Contingent Leg

The Contingent leg is trivially obtained from the above results. Similarly as $\mathcal{T}(1)$, we define $\mathcal{T}(k)$ as the set of dates containing $t_0$ and all the coupon dates between $t_0$ and $t_k^\star$. Hence, using the summation symbol $\sum_{\mathcal{T}(j)\setminus\mathcal{T}(j-1)}$ (with $j>1$) for meaning that the sum is taken over the coupon dates $t_{i-1}, t_i$ satisfying $t_{j-1}^\star \leqslant t_{i-1} < t_i \leqslant t_j^\star$, one gets

$$\begin{aligned}
\mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\} &= \mathrm{not}(n)(1-\mathrm{rec}(n)) \times \Bigg\{ p_1^\star \sum_{\mathcal{T}(1)} \frac{t_i - t_{i-1}}{t_1^\star - t_0}\delta(t_i) \\
&\quad + (p_2^\star - p_1^\star) \sum_{\mathcal{T}(2)\setminus\mathcal{T}(1)} \frac{t_i - t_{i-1}}{t_2^\star - t_1^\star}\delta(t_i) \\
&\quad + \ldots \\
&\quad + (p_{k-1}^\star - p_{k-2}^\star) \sum_{\mathcal{T}(k-1)\setminus\mathcal{T}(k-2)} \frac{t_i - t_{i-1}}{t_{k-1}^\star - t_{k-2}^\star}\delta(t_i) \\
&\quad + (p - p_{k-1}^\star) \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \frac{t_i - t_{i-1}}{t_k^\star - t_{k-1}^\star}\delta(t_i) \Bigg\} \ .
\end{aligned}$$

This expression can be simplified as

$$\begin{aligned}
\mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\} &= \mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\right\} \\
&\quad + (p - p_{k-1}^\star)\,\mathrm{not}(n)(1-\mathrm{rec}(n)) \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \frac{t_i - t_{i-1}}{t_k^\star - t_{k-1}^\star}\delta(t_i) \ .
\end{aligned} \tag{7}$$

Indeed, there is no reason that the parameters valid for a date $t_0 \leqslant t_i \leqslant t_1^\star$ when computing a CDS with maturity $t_1^\star$ are not the same, on the same date, when pricing a CDS with maturity date $t_2^\star > t_1^\star$ (the recovery, the notional, the coupon dates before $t_1^\star$ and the corresponding default probabilities must stay what they are). However, an additional term starts to play, explaining the difference between the two CDS prices.

### 2.2.2 Computation of the Fee Leg

Regarding the Fee leg, one has

$$\begin{aligned}
\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\} &= \mathrm{sp}_k(n,t_0)\,\mathrm{not}(n) \times \Bigg\{ \sum_{\mathcal{T}(1)} \left[\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - p_1^\star\left(\frac{t_i - t_0 - \Delta_i}{t_1^\star - t_0}\right)\right)\right] \\
&\quad + \sum_{\mathcal{T}(2)\setminus\mathcal{T}(1)} \left[\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - \left[p_1^\star + (p_2^\star - p_1^\star)\left(\frac{t_i - t_1^\star - \Delta_i}{t_2^\star - t_1^\star}\right)\right]\right)\right] \\
&\quad + \ldots \\
&\quad + \sum_{\mathcal{T}(k-1)\setminus\mathcal{T}(k-2)} \left[\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - \left[p_{k-2}^\star + (p_{k-1}^\star - p_{k-2}^\star)\left(\frac{t_i - t_{k-2}^\star - \Delta_i}{t_{k-1}^\star - t_{k-2}^\star}\right)\right]\right)\right] \\
&\quad + \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \left[\delta(t_i)\frac{t_i - t_{i-1}}{\gamma}\left(1 - \left[p_{k-1}^\star + (p - p_{k-1}^\star)\left(\frac{t_i - t_{k-1}^\star - \Delta_i}{t_k^\star - t_{k-1}^\star}\right)\right]\right)\right] \Bigg\} \ .
\end{aligned}$$

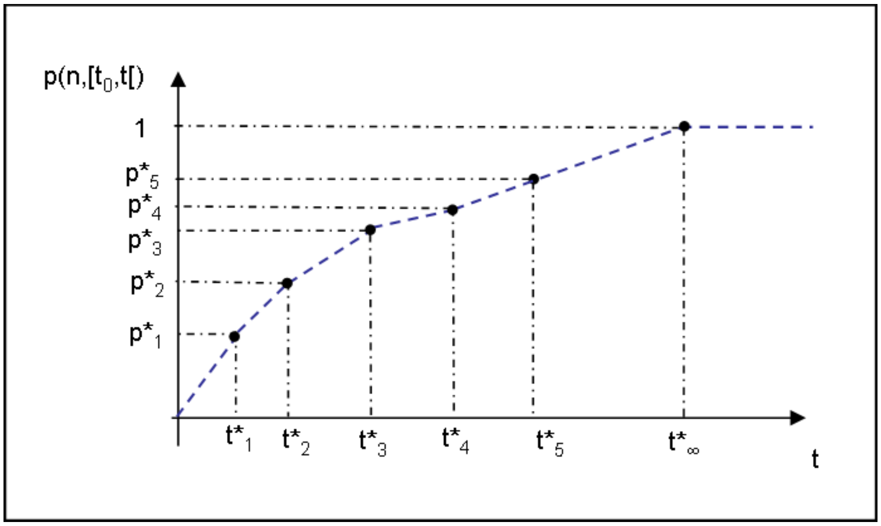



Figure 2: Evolution of the cumulative default probability of CDS on $n$ vs time (in this example, the number $m$ of available spreads is 5).

Few algebraic manipulation yields easily

$$\begin{aligned}\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\} &= \frac{\text{sp}_k(n,t_0)}{\text{sp}_{k-1}(n,t_0)}\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\right\} \\ &+\text{sp}_k(n,t_0)\,\text{not}(n)\times \\ &\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\left[\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(1-\left[p_{k-1}^\star+(p-p_{k-1}^\star)\left(\frac{t_i-t_{k-1}^\star-\Delta_i}{t_k^\star-t_{k-1}^\star}\right)\right]\right)\right]\ .\end{aligned} \tag{8}$$

In this leg, all the fees for the payment dates prior to $t_{k-1}^\star$ remain unchanged for a further maturity, except that the corresponding spreads changed from $\text{sp}_{k-1}(n,t_0)$ to $\text{sp}_k(n,t_0)$, which requires an adjustment. Further, a new term is involved to cover the period $[t_{k-1}^\star,t_k^\star[$.

#### 2.2.3 Computation of the cumulative Default probability $p_k^\star = p_k(n,[t_0,t^\star[)$

As usual, the probability that $n$ will default before $t_k^\star$ is given by $p$ such that the MtM is zero, i.e.:

$$\mathcal{N}\left\{\mathbb{E}[\text{CgtLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\} = \mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n,t_0,t_k^\star,\ldots),\ldots]\right\}$$

This specific value for $p$ is noted $p_k^\star$. Our cumulative default probability of $n$, $p_k(n,[t_0,t_k^\star[)$ will be set equal to $p_k^\star$.

Figure 2 shows the evolution of the (piecewise linear) cumulative default probability of $n$ vs time under the linearized model. We adopt the convention that $p(n,[t_0,t_0[)=0$, and that the cdf is linear in the time interval $[t_m^\star,t_\infty^\star[$ with $t_m^\star$ the last maturity date for which a spread is available; $t_\infty^\star$ is the threshold date before which we are sure, according to our linear model, that $n$ will default. The slope of the cumulative default probability of $n$ in $[t_m^\star,t_\infty^\star[$ is set as the same as the one in $[t_{m-1}^\star,t_m^\star[$. Obviously, $p(n,[t_\infty^\star,\infty[)=0$ so that $p(n,[t_0,t_\infty^\star[)=p(n,[t_0,\infty[)=1$.

Let us now compute the closed-form solution for $p_k^\star$ (remind that everything is linear in $p$). By using the assumption that the previous cumulative default probability was such that the MtM for the $k-1$-th maturity was zero, then

$$\mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\right\} = \mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\right\}$$

and, from eq.(3), (7) and (8), it is seen that $p_k^\star$ satisfies

$$\begin{aligned}(p_k^\star - p_{k-1}^\star) \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \frac{t_i - t_{i-1}}{t_k^\star - t_{k-1}^\star}\delta(t_i)(1-\mathrm{rec}(n))\,\mathrm{not}(n) = \\ \mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\}\frac{\mathrm{sp}_k(n,t_0)-\mathrm{sp}_{k-1}(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)} \\ +\,\mathrm{sp}_k(n,t_0)\,\mathrm{not}(n) \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \left\{\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left[1-\left(p_{k-1}^\star + (p_k^\star - p_{k-1}^\star)\frac{t_i - t_{k-1}^\star - \Delta_i}{t_k^\star - t_{k-1}^\star}\right)\right]\right\}\end{aligned}$$

i.e.

$$\boxed{p_k^\star = p_{k-1}^\star + \frac{\frac{\mathcal{N}\{E[FeeLeg(n,t_0,t_{k-1}^\star,\ldots),\ldots]\}}{(1-\mathrm{rec}(n))\,\mathrm{not}(n)}\frac{\mathrm{sp}_k(n,t_0)-\mathrm{sp}_{k-1}(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)} + \frac{\mathrm{sp}_k(n,t_0)}{1-\mathrm{rec}(n)}(1-p_{k-1}^\star)\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\frac{t_i-t_{i-1}}{t_k^\star-t_{k-1}^\star}\delta(t_i) + \frac{\mathrm{sp}_k(n,t_0)}{1-\mathrm{rec}(n)}\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(\frac{t_i-t_{k-1}^\star-\Delta_i}{t_k^\star-t_{k-1}^\star}\right)}}, \tag{9}$$

# 3 About the CDS reverting curves

In this section, the previously developed model will be used for analyzing whether a CDS curve can be arbitrarily reverted. In the first subsection, qualitative arguments will be given, showing that there must exist a threshold on the reversion intensity. Hopefully, this first step will convince the reader that the existence of such a threshold is very natural. In the second subsection, a rigorous theoretical development is proposed, yielding the analytical expression of this threshold.

## 3.1 Qualitative analysis

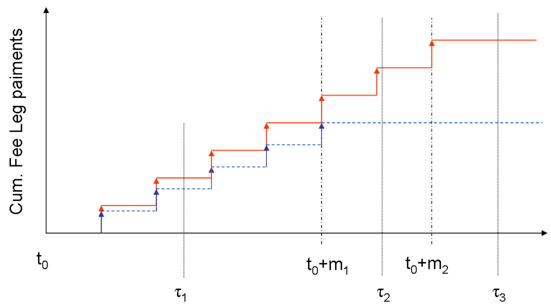
In order to understand the impact of the shape of CDS curve (increasing, constant, decreasing), and to prove that there must exist a threshold on the reversion intensity, we will compare four study cases. Let us consider two arbitrary maturities, $m_1, m_2$ satisfying $m_2 > m_1$. We will compare the Fee leg cash flows for the CDS with maturity $m_1$ and spread $\mathrm{sp}_{m_1}(n,t_0)$ to the CDS with maturity $m_2$ and a varying spread $\mathrm{sp}_{m_2}(n,t_0)$. This study will show that if we note $\mathrm{sp}_{m_2}(n,t_0) = \mathrm{sp}_{m_1}(n,t_0) - \xi$ where $\xi \geqslant 0$ then there exists an upper bound for $\xi$ lower than $\mathrm{sp}_{m_1}(n,t_0)$, i.e. a lower bound for $\mathrm{sp}_{m_2}(n,t_0)$ in order to avoid arbitrage opportunities.

### 3.1.1 Case 1: $\mathrm{sp}_{m_2}(n,t_0) > \mathrm{sp}_{m_1}(n,t_0)$

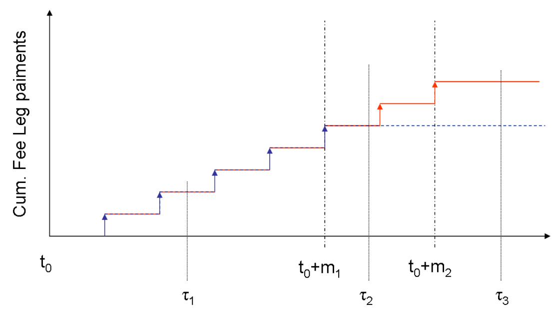
Assume a given default time $\tau$. Whatever is $\tau$ the total amount paid for a $m_2$-year protection buyer will be larger than for the $m_1$-year protection buyer. This larger price results from the higher credit risk. The evolution of the total payment made is plotted on Figure 3(a).

### 3.1.2 Case 2: $\mathrm{sp}_{m_2}(n,t_0) = \mathrm{sp}_{m_1}(n,t_0)$

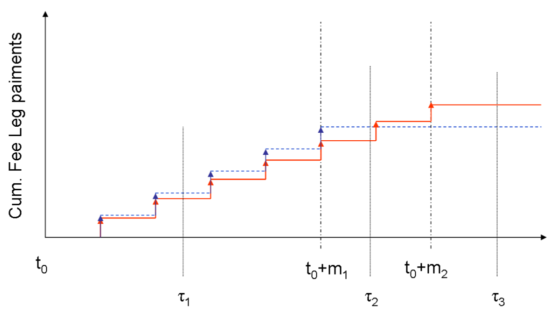
Assume a given default time $\tau$. Whatever is $\tau \leqslant t_0 + m_1$years, the total amount paid for a $m_2$-year protection buyer will be the same than for the $m_1$-year protection buyer. Whatever is $\tau > t_0 + m_1$years,



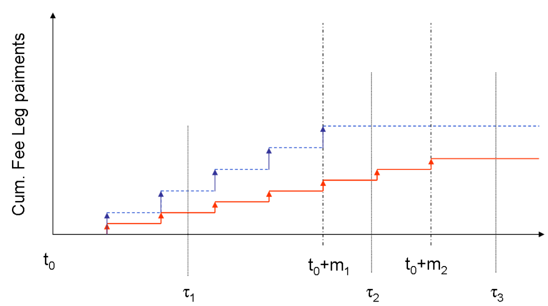

(a) Case 1




(b) Case 2




(c) Case 3




(d) Case 4

Figure 3: Four study cases. Evolution of the total Fee leg payments in time in case of no-default. These payments are made only at coupon dates. The continuous curve shows the $m_2$-year maturity CDS contract, the dashed curve the $m_1$-year ($m_1 < m_2$) maturity CDS contract. The length of the arrows is proportional to the value of the spread. The amounts corresponding to the intersection of these curves with the vertical dotted lines plotted at time $\tau_1, \tau_2, \tau_3$ indicate the total amount to pay in case of default at this time (note that, rigorously speaking, this amount has to be accrued by some small amount for the protection period offered between the default time $\tau$ and the previous coupon date, say $t_i$, which is $(\tau - t_i)/\gamma\,\mathrm{sp}_{m_i}(n, t_0)(1 - \mathrm{rec}(n))\,\mathrm{not}(n)$). The fourth case is very specific since the $m_2$ contract is deterministically more interesting than the $m_1$ one.

the total amount paid for a $m_2$-year protection buyer will be larger than for the $m_1$-year protection buyer. Again, these situations are very natural: you pay the same amount by unit of time in the two cases. The evolution of the total payment made is plotted on Figure 3(b).

#### 3.1.3 Case 3: $\mathrm{sp}_{m_2}(n,t_0) < \mathrm{sp}_{m_1}(n,t_0)$ but $\mathrm{sp}_{m_2}(n,t_0) \approx \mathrm{sp}_{m_1}(n,t_0)$

In this case, we assume that $\mathrm{sp}_{m_2}(n,t_0)$ is a bit lower than $\mathrm{sp}_{m_1}(n,t_0)$. This analysis aims at showing whether any value for $\mathrm{sp}_{m_2}(n,t_0)$ lower than $\mathrm{sp}_{m_1}(n,t_0)$ yields specific results. Assume a given default time $\tau$. Then, if $\tau < t_0 + m_1$ years the total amount paid for a $m_2$-year protection buyer will be smaller than for the $m_1$-year protection buyer. On the other hand, if $\tau > t_0 + m_1$ years $+\,\xi$ for some $\xi > 0$, then it is possible that the total amount paid for the $m_2$-year protection buyer becomes larger. Then, it may be interesting for a trader to choose either the $m_1$-year or the $m_2$-year CDS contract depending on the view he has on the default time. Hence, there is no reason for constraining the inequality $\mathrm{sp}_{m_2}(n,t_0) > \mathrm{sp}_{m_1}(n,t_0)$ to hold true in all the cases. The evolution of the total payment made is plotted on Figure 3(c).

#### 3.1.4 Case 4: $\mathrm{sp}_{m_2}(n,t_0) << \mathrm{sp}_{m_1}(n,t_0)$

Here, we assume a very low $\mathrm{sp}_{m_2}(n,t_0)$ compared to $\mathrm{sp}_{m_1}(n,t_0)$. If $\mathrm{sp}_{m_2}(n,t_0)$ is sufficiently small, then it can be seen that whenever is the default, the total amount paid by the $m_2$-year protection buyer will be smaller than the total amount paid by the $m_1$-year protection buyer. In other words, there is no advantage to buy the $m_1$-year protection contract knowing that such a $m_2$-year protection contract exists. Yet another way to see things is that there is an arbitrage opportunity. Assume a trader buys the $m_2$-year CDS contract and sells the $m_1$-year protection contract. With that position, the sign of the MtM of this position is no longer stochastic: our trader is sure to have a positive gain (but the amount remains stochastic). Furthermore, he will receive a protection against a credit event on the time period starting at $t_0 + m_1$ years and ending at $t_0 + m_2$ years for free. This is illustrated in Figure 3(d).

Our above qualitative analysis shows that there must be a threshold value on $\mathrm{sp}_{m_2}(n,t_0)$ depending on $\mathrm{sp}_{m_1}(n,t_0)$, whatever are the maturities $m_2 > m_1$ and $\mathrm{sp}_{m_1}(n,t_0)$. It cannot be arbitrarily lower than the spread $\mathrm{sp}_{m_1}(n,t_0)$.

### 3.2 Quantitative analysis

The question here is "Can the minimum value for $\mathrm{sp}_k(n,t_0)$ such that $p_k^\star \geqslant p_{k-1}^\star$ be lower than $\mathrm{sp}_{k-1}(n,t_0)$ ?" The previous qualitative analysis indicates that the answer is yes. Let us check that mathematically, and look for the analytical value of this lower bound on $\mathrm{sp}_k(n,t_0)$.

Since $p_k^\star$ denotes a cumulative default probability, it is obvious that $p_k^\star$ must satisfy $p_k^\star \geqslant p_{k-1}^\star$. However, from equation (9), it is clear that this condition might be violated if the relative spread differences $\frac{\mathrm{sp}_k(n,t_0)-\mathrm{sp}_{k-1}(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)}$ becomes so negative that the whole numerator becomes negative as well (the denominator is always positive). We call this phenomenon the *reverting curve* problem.

Let us first introduce the discussion with the following observation. Because the Contingent leg does not depend on the spread (see eq.(7)) and, on the other hand, the Fee leg is proportional to it (see eq.(8)), one gets that

$$p_k^\star \to p_{k-1}^\star - \frac{\mathcal{N}\left\{\mathbb{E}[\mathrm{CgtLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\right\}}{\mathrm{not}(n)(1-\mathrm{rec}(n)) \times \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \frac{t_i - t_{i-1}}{t_k^\star - t_{k-1}^\star}\delta(t_i)} \quad \text{as} \quad \mathrm{sp}_k(n,t_0) \to 0 \ .$$

Hence, because the second term of the $\lim_{\mathrm{sp}_k(n,t_0)\to 0} p_k^\star$ expression is strictly positive, there exists a threshold value $\mathrm{sp}_k^\star(n,t_0)$ for the $k$th maturity spread such that $p_k^\star < p_{k-1}^\star$ for all $\mathrm{sp}_k(n,t_0) \leqslant \mathrm{sp}_k^\star(n,t_0)$. Of course, this cannot happen since a CDF must be an increasing function. This behavior is illustrated on Figure 4.

In order to be more precise on the threshold values leading to that strange behavior, we split the above questions in the two following ones:

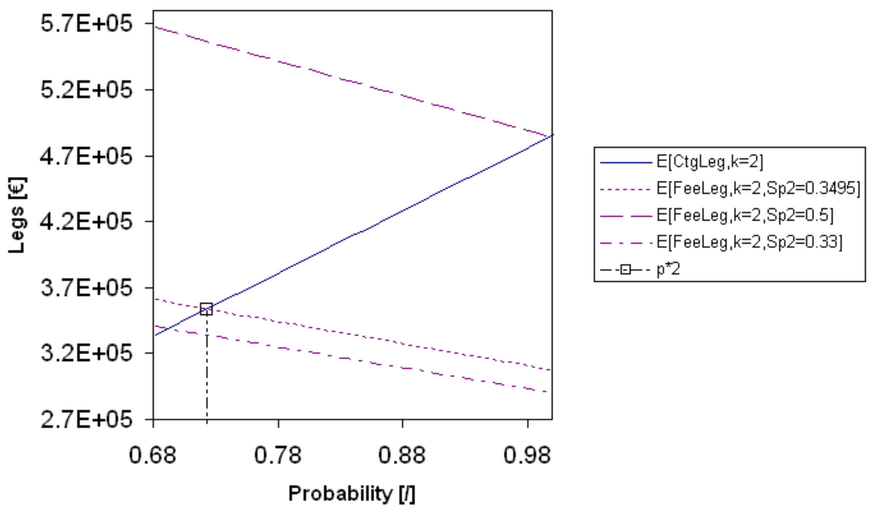



Figure 4: Evolution of the Contingent and Fee legs and the associated risk-neutral cumulative PD leading to a zero-MtM for various spread levels for $\mathrm{sp}_k(n, t_0) : 0 \leqslant s_1 \leqslant s_2 \leqslant s_3$. Example with $k = 2$ (corresponding to the numerical example given in Section 5.

1. How does the probability $p_k^\star$ vary with $\mathrm{sp}_k(n, t_0)$ ?
2. What is the minimum value of $\mathrm{sp}_k(n, t_0)$ $(k > 1)$ such that $p_k \geqslant p_{k-1}$ (which precisely corresponds to our threshold $\mathrm{sp}_k^\star(n, t_0)$)?

These questions will be addressed in the two following subsections.

#### 3.2.1 About the monotonic behavior of the curve $p_k^\star$ vs $\mathrm{sp}_k(n, t_0)$

Does the spread $\mathrm{sp}_k(n, t_0)$ vary monotonously with $p_k^\star$ on the interval $p_{k-1}^\star \leqslant p_k^\star \leqslant 1$ ? A natural answer is yes, but let's check that to give credit to our formula. Using the notation of Section 2.2.3 and eq.(9), one can see that the *implied fair spread* $\mathrm{sp}_k(n, t_0)$ equals

$$\frac{\frac{\mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\}}{(1-\mathrm{rec}(n))\,\mathrm{not}(n)} + \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)} \frac{t_i - t_{i-1}}{t_k^\star - t_{k-1}^\star}\delta(t_i)(p_k^\star - p_{k-1}^\star)}{\frac{(1-p_{k-1}^\star)\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{1-\mathrm{rec}(n)} + \frac{\mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t_{k-1}^\star,\ldots),\ldots]\}}{(1-\mathrm{rec}(n))\,\mathrm{not}(n)\,\mathrm{sp}_{k-1}(n,t_0)} - \frac{(p_k^\star - p_{k-1}^\star)\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(\frac{t_i - t_{k-1}^\star - \Delta_i}{t_k^\star - t_{k-1}^\star}\right)}{1-\mathrm{rec}(n)}}\ , \quad (10)$$

which is a function of the following form in $p_k^\star$ :

$$\mathrm{sp}_k(n, t_0) = \frac{a + b(p_k^\star - p_{k-1}^\star)}{c - d(p_k^\star - p_{k-1}^\star)}\ , \quad (11)$$

where $a, b, c, d$ and $p_k^\star - p_{k-1}^\star$ are positive numbers. This means that if $p_k^\star$ increases, so will do the numerator in the right-hand side of eq.(11) and its denomerator will decrease. Hence, as long as the above denominator is positive, $\mathrm{sp}_k(n, t_0)$ will be positive and monotonously increasing with $p_k^\star$. The threshold spread corresponding to $p_k^\star = 1$ is noted $\mathrm{sp}_k^\infty(n, t_0)$ (see the Figure 5 for an illustration).

To see that the spreads are always positive (i.e. to see that the above denominator is always positive) whatever is $p_{k-1}^\star \leqslant p_k^\star \leqslant 1$, it is sufficient to prove that $\mathrm{sp}_k^\infty(n, t_0) \geqslant 0$. Indeed, the $p_k^\star = 1$ condition yields

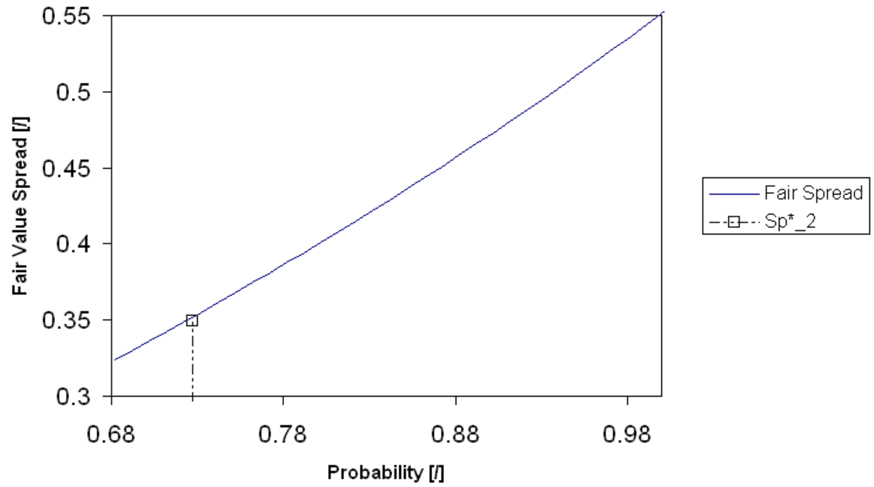



Figure 5: Evolution of the Fair Value spread $\mathrm{sp}_k(n,t_0)$ vs the probability $p^\star_k$ ranging in $p^\star_{k-1} \leqslant p^\star_k \leqslant 1$. The spread $\mathrm{sp}^\star_k(n,t_0)$ corresponds to $p^\star_k = p^\star_{k-1}$ and $\mathrm{sp}^\infty_k(n,t_0)$ corresponds to $p^\star_k = 1$ (curve given for illustration purposes only). The values are taken from the numerical example given in Section 5, with $k = 2$.

the lowest achievable bound for the above denominator: if we don't get a negative value in that case, we will never have a negative value for $\mathrm{sp}_k(n,t_0)$ under the constraint that $p^\star_{k-1} \leqslant p^\star_k \leqslant 1$. To show that, observe that

$$c \quad \geqslant \quad \frac{(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\backslash\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{1-\mathrm{rec}(n)} \quad ,$$

while

$$\begin{aligned} d(1-p^\star_{k-1}) \quad &= \quad \frac{\sum_{\mathcal{T}(k)\backslash\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\frac{t_i-t^\star_{k-1}-\Delta_i}{t^\star_k-t^\star_{k-1}}}{1-\mathrm{rec}(n)}(1-p^\star_{k-1}) \\ &< \quad \frac{(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\backslash\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{1-\mathrm{rec}(n)} \quad , \end{aligned}$$

where the above strict inequality results from that each of the terms $\frac{t_i-t^\star_{k-1}-\Delta_i}{t^\star_k-t^\star_{k-1}}$ is strictly lower than 1. This shows that for $p^\star_{k-1} \leqslant p^\star_k \leqslant 1$, the lowest possible value for $c - d(p^\star_k - p^\star_{k-1})$ is strictly positive, and so is the Fair value spread $\mathrm{sp}_k(n,t_0)$ in that probability range.

#### 3.2.2 On the reversion threshold of CDS curves

Let us now derive the threshold value for the spreads. From equation (9), the spread $\mathrm{sp}^\star_k(n,t_0)$ satisfies

$$\frac{\mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{\mathrm{not}(n)}\frac{\mathrm{sp}^\star_k(n,t_0)-\mathrm{sp}_{k-1}(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)} + \mathrm{sp}^\star_k(n,t_0)(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\backslash\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma} = 0 \ ,$$

i.e.

$$\frac{\mathcal{N}\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{\mathrm{not}(n)}\Big(1-\frac{\mathrm{sp}^\star_k(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)}\Big) = \mathrm{sp}^\star_k(n,t_0)(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\backslash\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma} \ .$$

Rearranging these terms leads quickly to

$$\boxed{\mathrm{sp}^\star_k(n,t_0) = \frac{\mathrm{sp}_{k-1}(n,t_0)}{1+\frac{\mathrm{sp}_{k-1}(n,t_0)\,\mathrm{not}(n)(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\}}}\quad .} \tag{12}$$

The above equation confirms, as expected from the qualitative analysis, that i) there exists a minimum value $\mathrm{sp}^\star_k(n,t_0)$ for $\mathrm{sp}_k(n,t_0)$ such that probability axioms are met, and ii) that this threshold value can be (and actually is always) lower than $\mathrm{sp}_{k-1}(n,t_0)$, i.e. any CDS curve may be decreasing.

Actually, the qualitative analysis performed in Section 7.4 already showed that such a threshold should exist in order to avoid a *sure* (i.e. deterministic) arbitrage. Let us now turn to the interpretation of the above analytical threshold value from a business perspective. As it will be shown below, there is no redundancy between the qualitative and quantitative results.

# 4 On the business meaning of the threshold value $\mathrm{sp}^\star_k(n,t_0)$

To answer the question "what is the meaning of this threshold value on spreads", we will tackle the problem upside down. Let us first answer the question "What is the specific value $\mathrm{sp}^{\star\star}_k(n,t_0)$ of the spread $\mathrm{sp}_k(n,t_0)$ such that the NPV of the $\mathrm{FeeLeg}(n,t_0,t^\star_k,\ldots)$ equals those of $\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots)$ (and similarly for the Contingent Leg as the MtM is zero) ?"

By definition, $\mathrm{sp}^{\star\star}_k(n,t_0)$ is the value of the spread $\mathrm{sp}_k(n,t_0)$ such that

$$\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\} = \mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k},\ldots),\ldots]\right\} \quad ,$$

meaning with that spread, you have a period of $\frac{t^\star_k-t^\star_{k-1}}{\gamma}$ years of protection for free as a protection buyer ?

The spread $\mathrm{sp}^{\star\star}_k(n,t_0)$ satisfies

$$\begin{aligned}
\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\} &= \mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k},\ldots),\ldots]\right\} \\
&= \frac{\mathrm{sp}^{\star\star}_k(n,t_0)}{\mathrm{sp}_{k-1}(n,t_0)}\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\} \\
&\quad + \mathrm{sp}^{\star\star}_k(n,t_0)\,\mathrm{not}(n)\times \\
&\quad \sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\left[\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(1-\left[p^\star_{k-1}+(p-p^\star_{k-1})\frac{t_i-t^\star_{k-1}-\Delta_i}{t^\star_k-t^\star_{k-1}}\right]\right)\right] \quad .
\end{aligned}$$

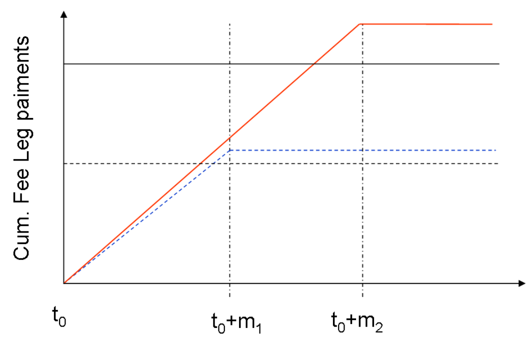
Applying consecutively basic algebraic relationships (see the Appendix 7.3 for more detailed calculations), one gets

$$\boxed{\mathrm{sp}^{\star\star}_k(n,t_0) = \frac{\mathrm{sp}_{k-1}(n,t_0)}{1+\frac{\mathrm{sp}_{k-1}(n,t_0)\,\mathrm{not}(n)(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\}}}\quad .} \tag{13}$$

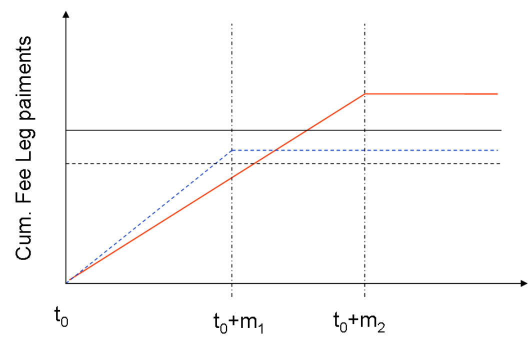
Hence, we obtain the remarkable result that

$$\boxed{\mathrm{sp}^\star_k(n,t_0) = \mathrm{sp}^{\star\star}_k(n,t_0)\quad .} \tag{14}$$

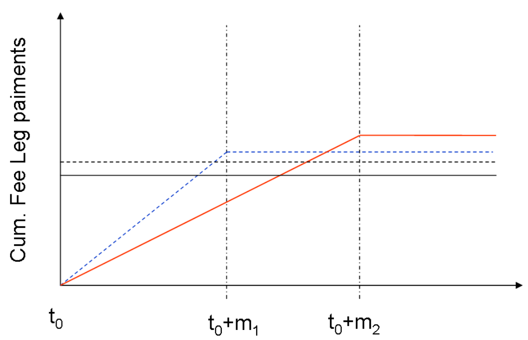
In other words, the spread $\mathrm{sp}^\star_k(n,t_0)$ corresponding to $p^\star_k = p^\star_{k-1}$ (the lowest $k$-th maturity spread such that the cumulative default probability function is not decreasing between $t^\star_{k-1}$ and $t^\star_k$) equals the spread $\mathrm{sp}^{\star\star}_k(n,t_0)$, which gives a lower bound for $\mathrm{sp}_k(n,t_0)$ such that the reverted curve does not imply arbitrage opportunities as understood by the model, which is working with (discounted) expected values. This is illustrated on Figure 6.



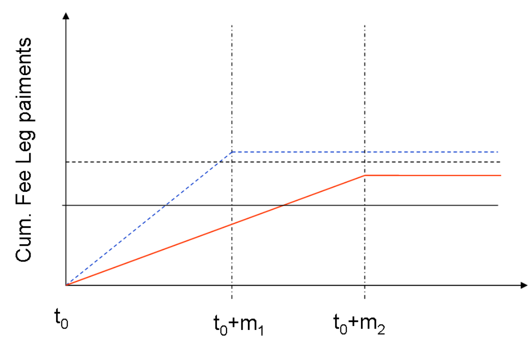

(a) $\mathrm{sp}_k(n, t_0) \geqslant \mathrm{sp}_{k-1}(n, t_0)$ and $\mathrm{sp}_k(n, t_0) > \mathrm{sp}_k^{\star}(n, t_0)$ : no expected arbitrage




(b) $\mathrm{sp}_k(n, t_0) \leqslant \mathrm{sp}_{k-1}(n, t_0)$ and $\mathrm{sp}_k(n, t_0) > \mathrm{sp}_k^{\star}(n, t_0)$ : no expected arbitrage




(c) $\mathrm{sp}_k(n, t_0) < \mathrm{sp}_k^{\star}(n, t_0)$ : expected arbitrage




(d) $\mathrm{sp}_k(n, t_0) < \mathrm{sp}_k^{\star}(n, t_0)$ and total no-default Fee leg payment of $k$-th maturity smaller than total no-default Fee leg payment of $k-1$-th maturity : certain arbitrage

Figure 6: Evolution of the no-default payment curve of the Fee leg in time (increasing, capped curves) for the $k-1$th (dashed) and $k$-th (solid) maturities, and discounted expected value of Fee leg payments (horizontal dashed and solid curves, respectively). The curves have been smoothed so that the daily step structure due to our discrete day-by-day time scale are no longer visible. Arbitrage or no-arbitrage in expected value depending on the spread $\mathrm{sp}_k(n, t_0)$ with respect to the threshold value $\mathrm{sp}_k^{\star}(n, t_0)$. Certain arbitrage depends on the total no-default Fee leg payment.

Consequently, while the previous development gives a threshold value for avoiding an arbitrage *in expected value*, the qualitative analysis only showed that the threshold value preventing sure arbitrage must exist. A similar development than the one performed above shows that, as expected, this "*no arbitrage in expected value* threshold" is more restrictive than the "*certain arbitrage* one". In other words, the qualitative analysis explains a part of the problem, but issues might still occur even if no sure arbitrage is observed: there is no need that the CDS curve yields a certain arbitrage for blocking the CDS pricer (which is dealing with discounted expected values). Hence, even CDS curves not yielding sure arbitrage might be blocking. The effective threshold is more restrictive because it prevents arbitrage in expected values as well.

# 5 Numerical example

We shall first consider a "toy" example in which only few computations are involved. Next, the real-life example of CDS on *Residential Capital LLC* corresponding to the dashed CDS curve in Figure 1 will be considered, showing that real-life CDS curves might yield to a blocking of the pricer.

## 5.1 Toy example

We consider a 3-maturity deal (that is three tenors are available for the underlying name, $m = 3$) on a fake name $n$ with the maturity dates 22/12/2008 (one-year CDS), 21/12/2009 (two-year CDS) and 20/12/2010 (three-year CDS). The contract is assumed to start at the MtM date, $t_0 = 15/11/2007$. In order that credit events occurring during the day corresponding to the maturity dates are covered, we have, according to our convention, to increase them by one day in our pricing model (this is because we assumed for the simplicity of the equations that the maturity date was excluded from the protection duration). We assume that $\gamma = 360$, $\text{sp}_1(n,t_0) = 459.7871$, $\text{sp}_2(n,t_0) = 349.5433$, $\text{sp}_3(n,t_0) = 366.8158$. The involved coupon dates and associated discount factors are shown on Table 1. We used the assumption that if a name defaults between two coupon dates, then, in expectation, the default will occur in the middle of the time interval: $\Delta_i = (t_i - t_{i-1})/2$, which is consistent with our piecewise linear model for cumulative default probabilities, as shown in Appendix 7.2.

| label | $t_i$ | $\delta(t_i)$ | $(t_i - t_{i-1})/\gamma$ |
|---|---|---|---|
| $t_0$ | 15/11/2007 | 1 | - |
| $t_1$ | 20/12/2007 | 0.9971 | 0.0972 |
| $t_2$ | 20/03/2008 | 0.9896 | 0.2528 |
| $t_3$ | 20/06/2008 | 0.9820 | 0.2556 |
| $t_4$ | 22/09/2008 | 0.9743 | 0.2611 |
| $t_5 = t_1^\star$ | 23/12/2008 | 0.9669 | 0.2528 |
| $t_6$ | 20/03/2009 | 0.9599 | 0.2417 |
| $t_7$ | 22/06/2009 | 0.9524 | 0.2611 |
| $t_8$ | 21/09/2009 | 0.9452 | 0.0.2528 |
| $t_9 = t_2^\star$ | 22/12/2009 | 0.9380 | 0.2556 |
| $t_{10}$ | 22/03/2010 | 0.9341 | 0.2500 |
| $t_{11}$ | 21/06/2010 | 0.9270 | 0.2528 |
| $t_{12}$ | 20/09/2010 | 0.9200 | 0.2528 |
| $t_{13} = t_3^\star$ | 21/12/2010 | 0.9130 | 0.2556 |

Table 1: Input data of our numerical example (rounded to fourth digit after the coma.)

From the equations developed in this paper, one gets the following results:

In the previous sections, we have developed a procedure to find the maximum level of reversion for a CDS curve. The two threshold spreads are, according to our equations : $\text{sp}_2^\star(n,t_0) \approx 251.6348$ and $\text{sp}_3^\star(n,t_0) \approx$

| | $k=1$ | $k=2$ | $k=3$ |
|---|---|---|---|
| Fee Leg | 48087.7559 | 66162.436284 | 97846.7593 |
| Contingent Leg | 48087.7559 | 66162.436284 | 97846.7593 |
| $p_k^\star = \mathbb{P}(\boldsymbol{\tau} \leqslant t_k^\star)$ | 0.0982 | 0.1363 | 0.2049 |

Table 2: Results: the default probabilities $p_k^\star$ lead to a Mark-to-Market being zero up to the 15-th digit using Microsoft Excel (here: number rounded to the fourth digit after the coma).

245.1053. With these values instead of the original spreads ($\mathrm{sp}_2(n,t_0) \leftarrow 251.6348$, $\mathrm{sp}_3(n,t_0) \leftarrow 245.1053$), we find the following probabilities: $p_1^\star = p_2^\star \approx 0.0982$, $p_3^\star \approx 0.1391$. With this specif value for $\mathrm{sp}_2(n,t_0)$, we have yet another value for the third spread : $\mathrm{sp}_3^\star(n,t_0) \approx 174.6668$. The set of spreads $\mathrm{sp}_1(n,t_0) \leftarrow 0.04598$ , $\mathrm{sp}_2(n,t_0) \leftarrow 251.6348$ and $\mathrm{sp}_3(n,t_0) \leftarrow 174.6668$ would lead to a constant cumulative probability function on the $[t_1^\star, t_3^\star]$ time interval : $p_1^\star = p_2^\star = p_3^\star \approx 0.0982$, which proves the consistency of our equations.

## 5.2 Real-life example

In this section, we will compute the cumulative default probability curve from the dashed CDS curve shown in Figure 1. This curve corresponds to the CDS spreads of *Residential Capital LLC* on 2007 August 20th. Assume we are on that date and that we want to infer the cumulative default probability curve from this set of market quotes. It will be shown that this CDS curve is not *bootstrappable*, meaning that it is not possible to get a consistent cumulative probability curve using our model. Therefore, let us focus on the three first maturities: the 1-year, 2-year and 3-year spreads with spreads equal to 2840.7 bps, 2531.2 and 1691.8. The corresponding maturity dates are 22/092008, 21/092010 and 20/09/2012. The recovery rate is 40% and the number of days in a year is $\gamma = 360$. The discount factors $\delta(t_i)$ are given by $\mathrm{e}^{-r\frac{t_i - t_0}{\gamma}}$ with $r = 3\%$.

Recall that, because the market convention assumes that the protection period also includes the maturity date while, in our equations, it was not the case: we have to increase the maturity dates by one day, in order to price the related deals correctly with our framework ($t_i^\star \leftarrow t_i^\star + 1$).

| label | $t_i$ | $\delta(t_i)$ | $(t_i - t_{i-1})/\gamma$ |
|---|---|---|---|
| $t_0$ | 20/08/2007 | 1 | - |
| $t_1$ | 20/09/2007 | 0.9974 | 0.0861 |
| $t_2$ | 20/12/2007 | 0.9899 | 0.2528 |
| $t_3$ | 20/03/2008 | 0.9824 | 0.2528 |
| $t_4$ | 20/06/2008 | 0.9749 | 0.2556 |
| $t_5 = t_1^\star$ | 23/09/2008 | 0.9672 | 0.2639 |
| $t_6$ | 22/12/2008 | 0.9599 | 0.2500 |
| $t_7$ | 20/03/2009 | 0.9529 | 0.2444 |
| $t_8$ | 22/06/2009 | 0.9455 | 0.2611 |
| $t_9 = t_2^\star$ | 22/09/2009 | 0.9383 | 0.2556 |
| $t_{10}$ | 21/12/2009 | 0.9341 | 0.2500 |
| $t_{11}$ | 22/03/2010 | 0.9270 | 0.2528 |
| $t_{12}$ | 21/06/2010 | 0.9200 | 0.2528 |
| $t_{13} = t_3^\star$ | 21/09/2010 | 0.9130 | 0.2556 |

Table 3: Input data of our real-life numerical example (rounded to fourth digit after the coma.)

From the equations developed in this paper, one gets the results shown in Table 4 (notional is $1000,000$). The obtained cumulative default probabilities curve does not consist in an increasing sequence, which is not consistent with probability theory. This is because the CDS spread curve is too reverted given our set of assumptions and conventions. Our spread thresholds are $\mathrm{sp}_2^\star(n,t_0) \approx 1723.4584$ bps and $\mathrm{sp}_3^\star(n,t_0) \approx 1962.7514$

bps, which is larger than $\mathrm{sp}_3^\star(n, t_0) = 1691.8$ Therefore, the 3-year spread is below the corresponding spread threshold, and the pricer should block in that case.

| | $k = 1$ | $k = 2$ | $k = 3$ |
|---|---|---|---|
| Fee Leg | 245169.1097 | 339954.5813 | 298855.7416 |
| Contingent Leg | 245169.1097 | 339954.5813 | 298855.7416 |
| $p_k^\star = \mathbb{P}(\boldsymbol{\tau} \leqslant t_k^\star)$ | 0.4170 | 0.5834 | 0.5093 |

Table 4: Results: the default probabilities $p_k^\star$ lead to a Mark-to-Market being zero up to the 15-th digit using Microsoft Excel (here: number rounded to the fourth digit after the coma).

# 6 Conclusions

In this paper, a very simple model for pricing Credit Default Swap products has been proposed. Exactly as more involved models do, our approach relies on a given family of functions for the cumulative probability function, which is the piecewise linear function. The proposed model does not require additional parameters to tune, like e.g. hazard rate; all the remaining values are inferred by the market. However, in spite of its simplicity, this model fits pretty well to the market because it involves practical conventions, and that only few assumptions have been made. If desired, this model could still benefit from an additional degree of freedom, which can either be implied by the model, or can be tuned to come up with best and worst case fees. For the rest, up to the assumptions clearly stated in the introduction, the model is exact. Even though quite efficient, the purpose of our model is not really to compete with other existing ones in the academic literature. The main advantage of this model is its manageability: the equations, even though quite lengthy, are simple to manipulate, so that a lot of analytical expressions are easily available, like e.g. default probabilities, fair spreads, Mark-to-Market and sensitivities. Consequently, the model suits pretty well for understanding and to studying unexpected behaviors of CDS pricing engines. As indicated in the title of this report, this goal was the main motivation for developing the piecewise linear model.

In this analysis, we decided to focus on the so-called *reverted CDS curve* issue, which refers to the possible CDS pricer blockings resulting from the fact that the spread for, say, a 5-year CDS contract is "sufficiently lower" than the one applying for a 3-year contract. Analysing this problem was the second aim of this report. A qualitative development has been suggested to convince the reader that a spread threshold must exist, otherwise arbitrage opportunities may come into play. A deeper theoretical study confirms that statement, and further gives an analytical formula for this threshold. Finally, a business meaning of this threshold value has been given in terms of arbitrage opportunities in expected value. One consequence of this result is that the $k-1$-th maturity contract would never be entered knowing the $k$-th maturity contract when strongly reverted CDS curves are observed (the only reason for entering that contract would be based on possible liquidity issues, resulting from a non-rational behavior of the market, and that are not taken into account in our pricing model).

# 7 Appendix

## 7.1 NPV of the expected Fee leg

Hereafter, we justify the $i$-th term in the sum in the right-hand side of expression :

$$\boxed{\mathcal{N}\left\{\mathbb{E}[\mathrm{FeeLeg}(n, t_0, t_1^\star, \ldots), \ldots]\right\} = \mathrm{sp}_1(n, t_0)\,\mathrm{not}(n) \sum_{i=1}^{N'} \left[\delta(t_i) \frac{t_i - t_{i-1}}{\gamma} \left(1 - p\left(\frac{t_i - t_0 - \Delta_i}{t_1^\star - t_0}\right)\right)\right] \quad .}$$

Remind that we make the hypothesis that cash flows can only occur at coupon dates. This has the following consequence: the Fee leg payments are made at any coupon date coming before the maturity and before the possible default time. In case of default, the accrual payment due to the protection received for the period between the default time and the previous coupon date will be paid on the first coupon date after the default time. For the Contingent leg, the payment is made on the first coupon date after the default time, provided that a default occurs before the maturity date. In both cases, this means that only the discount factors at those coupon dates will be used.

The term we want to focus on is

$$\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(1-p\left(\frac{t_i-t_0-\Delta_i}{t_1^\star-t_0}\right)\right)$$

and represents the expected (Fee Leg) payments related to the time interval $[t_{i-1},t_i[$. To prove this, let us detail the three possible scenarios:

1. the default occurs before $t_{i-1}$ ($\tau < t_{i-1}$);
2. the default occurs before $t_i$ but after $t_{i-1}$ ($t_{i-1} \leqslant \tau < t_i$);
3. the default occurs after $t_i$ ($\tau \geqslant t_i$).

In the first case, the amount to pay is zero: the contribution of that event in the whole expectation is

$$0 \times \sum_{\tau<t_{i-1}} \{\mathbb{P}(\boldsymbol{\tau}=\tau)\} = 0 \ .$$

In the third case, the payment for the whole period between the consecutive coupon dates $t_{i-1}$ and $t_i$ has to be paid at $t_i$, so that the NPV of the payment is $\frac{\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)}{\gamma}\sum_{\tau\geqslant t_i}\{(t_i-t_{i-1})\,\mathbb{P}(\boldsymbol{\tau}=\tau)\}$ which reduces to

$$\frac{\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)}{\gamma}(t_i-t_{i-1})\,\mathbb{P}(\boldsymbol{\tau}\geqslant t_i) \ .$$

Finally, in the second case, the payment is $\frac{\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)}{\gamma}\sum_{t_{i-1}\leqslant\tau<t_i}(\tau-t_{i-1})\,\mathbb{P}(\boldsymbol{\tau}=\tau)$. By using conditioning, this last probability can be rewritten as

$$\mathbb{P}(\boldsymbol{\tau}=\tau|t_{i-1}\leqslant\boldsymbol{\tau}<t_i)\,\mathbb{P}(t_{i-1}\leqslant\boldsymbol{\tau}<t_i) \ ,$$

because $\mathbb{P}(\boldsymbol{\tau}=\tau|\overline{t_{i-1}\leqslant\boldsymbol{\tau}<t_i})\,\mathbb{P}(\overline{t_{i-1}\leqslant\boldsymbol{\tau}<t_i})$ (the overline denotes the complementary event) has a zero contribution in the above sum since $\mathbb{P}(\boldsymbol{\tau}=\tau|\overline{t_{i-1}\leqslant\boldsymbol{\tau}<t_i})=0$ for any $\tau$ satisfying $t_{i-1}\leqslant\tau<t_i$. Consequently, the contribution of the second case to the NPV of the expected Fee leg payment reduces to

$$\frac{\mathrm{sp}_1(n,t_0)\,\mathrm{not}(n)\delta(t_i)}{\gamma}\underbrace{\sum_{\tau}(\tau-t_{i-1})\,\mathbb{P}(\boldsymbol{\tau}=\tau|t_{i-1}\leqslant\boldsymbol{\tau}<t_i)}_{\Delta_i}\,\mathbb{P}(t_{i-1}\leqslant\boldsymbol{\tau}<t_i)$$

## 7.2 Computation of implied $\Delta_i$

A value for $\Delta_i$ is implied by our model. By definition, and using Bayes' rule, one gets

$$\begin{aligned}\Delta_i &= \mathbb{E}[\boldsymbol{\tau} - t_{i-1} | t_{i-1} \leqslant \boldsymbol{\tau} < t_i] \\ &= \sum_{\tau} (\tau - t_{i-1})\, \mathbb{P}(\boldsymbol{\tau} = \tau | t_{i-1} \leqslant \boldsymbol{\tau} < t_i) \\ &= \sum_{t_{i-1} \leqslant \tau < t_i} (\tau - t_{i-1}) \frac{\mathbb{P}(\boldsymbol{\tau} = \tau \,;\, t_{i-1} \leqslant \boldsymbol{\tau} < t_i)}{\mathbb{P}(t_{i-1} \leqslant \boldsymbol{\tau} < t_i)} \\ &= \frac{\sum_{t_{i-1} \leqslant \tau < t_i} (\tau - t_{i-1})\, \mathbb{P}(\boldsymbol{\tau} = \tau)}{\sum_{t_{i-1} \leqslant \tau < t_i} \mathbb{P}(\boldsymbol{\tau} = \tau)}\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathbb{P}(\boldsymbol{\tau} = \tau) &= \underbrace{\left\{ p^\star_{k-1} + (p^\star_k - p^\star_{k-1}) \frac{(\tau+1) - t^\star_{k-1}}{t^\star_k - t^\star_{k-1}} \right\}}_{\mathbb{P}(\boldsymbol{\tau} < \tau + 1 \text{ day}) = \mathbb{P}(\boldsymbol{\tau} \leqslant \tau)} - \underbrace{\left\{ p^\star_{k-1} + (p^\star_k - p^\star_{k-1}) \frac{\tau - t^\star_{k-1}}{t^\star_k - t^\star_{k-1}} \right\}}_{\mathbb{P}(\boldsymbol{\tau} < \tau)} \\ &= (p^\star_k - p^\star_{k-1}) \frac{(\tau+1) - \tau}{t^\star_k - t^\star_{k-1}} \\ &= \frac{p^\star_k - p^\star_{k-1}}{t^\star_k - t^\star_{k-1}} \ ,\end{aligned}$$

and this leads to

$$\begin{aligned}\Delta_i &= \frac{\sum_{t_{i-1} \leqslant \tau < t_i} (\tau - t_{i-1}) \times 1}{\sum_{t_{i-1} \leqslant \tau < t_i} \times 1} \\ &= \frac{\sum_{j=0}^{N_i} j}{\sum_{j=1}^{N_i} 1}\end{aligned}$$

where $N_i$ is the number of days between $t_{i-1}$ (included) and $t_i$ (excluded). Finally, we get

$$\Delta_i = \frac{\frac{N_i(N_i+1)}{2}}{N_i} \approx \frac{t_i - t_{i-1}}{2} \ .$$

## 7.3 Detailed calculation

For shorting equations, define $\nu_i \doteq \frac{t_i - t^\star_{k-1} - \Delta_i}{t^\star_k - t^\star_{k-1}}$. Then:

$$\begin{aligned}\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n, t_0, t^\star_{k-1}, \ldots), \ldots]\right\} &= \mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n, t_0, t^\star_k, \ldots), \ldots]\right\} \\ &= \frac{\text{sp}^{\star\star}_k(n, t_0)}{\text{sp}_{k-1}(n, t_0)} \mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n, t_0, t^\star_{k-1}, \ldots), \ldots]\right\} \\ &\quad + \text{sp}^{\star\star}_k(n, t_0)\, \text{not}(n) \times \\ &\qquad \sum_{\mathcal{T}(k) \setminus \mathcal{T}(k-1)} \left[ \delta(t_i) \frac{t_i - t_{i-1}}{\gamma} \left( 1 - \left[ p^\star_{k-1} + (p^\star_k - p^\star_{k-1}) \nu_i \right] \right) \right] \ .\end{aligned}$$

i.e. from eq.( 9)

$$\frac{\text{sp}_{k-1}(n, t_0) - \text{sp}^{\star\star}_k(n, t_0)}{\text{sp}_{k-1}(n, t_0)} \frac{\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n, t_0, t^\star_{k-1}, \ldots), \ldots]\right\}}{\text{sp}^{\star\star}_k(n, t_0)\, \text{not}(n)}$$

equals

$$\frac{(1-p^\star_{k-1})\displaystyle\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}-\left(\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i\right)\times \frac{\mathcal{N}\{\mathbb{E}[\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{(1-\text{rec}(n))\,\text{not}(n)}\frac{\text{sp}^{\star\star}_k(n,t_0)-\text{sp}_{k-1}(n,t_0)}{\text{sp}_{k-1}(n,t_0)}+\frac{\text{sp}^{\star\star}_k(n,t_0)}{1-\text{rec}(n)}(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}}{\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\frac{t_i-t_{i-1}}{t^\star_k-t^\star_{k-1}}\delta(t_i)+\frac{\text{sp}^{\star\star}_k(n,t_0)}{1-\text{rec}(n)}\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i}\ .$$

Equivalently,

$$\frac{\text{sp}_{k-1}(n,t_0)-\text{sp}^{\star\star}_k(n,t_0)}{\text{sp}_{k-1}(n,t_0)}\frac{\{\mathbb{E}[\mathcal{N}\,\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{\text{sp}^{\star\star}_k(n,t_0)\,\text{not}(n)}\times \left\{\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\frac{t_i-t_{i-1}}{t^\star_k-t^\star_{k-1}}\delta(t_i)+\frac{\text{sp}^{\star\star}_k(n,t_0)}{1-\text{rec}(n)}\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i\right\} \tag{15}$$

equals

$$(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\frac{t_i-t_{i-1}}{t^\star_k-t^\star_{k-1}}\delta(t_i)+\frac{\text{sp}^{\star\star}_k(n,t_0)}{1-\text{rec}(n)}\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i\right) -\left(\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i\right)\times \left\{\frac{\mathcal{N}\{\mathbb{E}[\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{(1-\text{rec}(n))\,\text{not}(n)}\frac{\text{sp}^{\star\star}_k(n,t_0)-\text{sp}_{k-1}(n,t_0)}{\text{sp}_{k-1}(n,t_0)}+\frac{\text{sp}^{\star\star}_k(n,t_0)}{1-\text{rec}(n)}(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\right\}\ ,$$

and, the last hand-side, after some simplifications, yields.

$$(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\left(\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\frac{t_i-t_{i-1}}{t^\star_k-t^\star_{k-1}}\delta(t_i)\right) -\left(\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\nu_i\right)\left\{\frac{\mathcal{N}\{\mathbb{E}[\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\}}{(1-\text{rec}(n))\,\text{not}(n)}\frac{\text{sp}^{\star\star}_k(n,t_0)-\text{sp}_{k-1}(n,t_0)}{\text{sp}_{k-1}(n,t_0)}\right\}\ . \tag{16}$$

By equating eq.(15) and eq.(16), one gets

$$\frac{\text{sp}_{k-1}(n,t_0)-\text{sp}^{\star\star}_k(n,t_0)}{\text{sp}_{k-1}(n,t_0)}\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\}=\text{sp}^{\star\star}_k(n,t_0)\,\text{not}(n)(1-p^\star_{k-1})\sum_{\mathcal{T}(k)\setminus\mathcal{T}(k-1)}\delta(t_i)\frac{t_i-t_{i-1}}{\gamma}\ ,$$

leading to eq.(13).

## 7.4 Reversion threshold in a certain arbitrage case

There is a certain arbitrage if, whenever occurs the possible default, the Fee leg for the $k$-th maturity is less than the $k-1$ th maturity one. Because the Fee leg is linear in time (see Figure 6), there is certain arbitrage if and only if the no default value of the $k$-th maturity Fee leg is lower than the no-default value of the $k-1$-th one. This leads to a threshold value (lower bound) for $\text{sp}_k(n,t_0)$ given $\text{sp}_{k-1}(n,t_0)$, noted $\text{sp}^\dagger_k(n,t_0)$.

In the certain arbitrage case, the threshold value has the same form than $\text{sp}^\star_k(n,t_0)$ but where

$$\mathcal{N}\left\{\mathbb{E}[\text{FeeLeg}(n,t_0,t^\star_{k-1},\ldots),\ldots]\right\}/(1-p^\star_k)$$

is replaced by the no-default value of the payment for the $k$-th maturity. On the other hand, because the no default value of the Fee leg payments are obtained using the same formula than the expected Fee leg when setting probabilities ($p$ and $p^\star_k$) to zero, quick algebraic manipulations show that $(1-p^\star_k)$ times the no-default value of the payments is lower than the NPV of the expected Fee leg, so that one obtains $\text{sp}^\dagger_k(n,t_0)<\text{sp}^\star_k(n,t_0)$. Hence, the no arbitrage in expected value yields a stronger condition in terms of maximum reversion level.[Cariboni (2004)]

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