Analytical Pricing of Basket Default Swaps in a Dynamic Hull & White Framework

Frédéric D. Vrins*

This Version : January 2010

Abstract

In this paper, some analytical results related to the Hull & White dynamic model of credit portfolio of N obligors in the case of constant jump size are provided. For instance, this specific assumption combined with the moment generating function of the Poisson process lead to analytical calibration for the model with respect to the underlying CDSs. Further, extremely simple analytical expressions are obtained for first-to-default swaps; the more general case of quantities related to nth-to-default swaps also have a closed form and remain tractable for small n. Similarly, pairwise correlation between default indicators also proves to be simple. Although the purpose of this note is not to compare models, we compare the shape of pairwise default correlations of the Hull & White, the Gaussian copula and the Mai & Scherer model with compound Poisson process as Lévy subordinator. It is shown that only the models including jumps can lead to non-vanishing default correlation for short-term maturities. Further, these models can generate higher default correlation levels compared to the Gaussian one. When calibrated on default probability of first default time, Jump-based models also lead to much higher default probability for the last obligor to default. Finally, we tackle the problem of simultaneous jumps, which prevent the above class of models to be usable when recoveries are name-specific. To that end, we propose a tractable compromise to deal with baskets being non-homogeneous recovery-wise under the Hull & White model by splitting isolated and non-isolated default events.

1 Introduction

As form the Summer 2007 credit crisis, a lot of researchers both from top universities and professional institutions try to develop a dynamic alternative to the (static) Gaussian copula model for the pricing of multi-underlying credit derivatives. The challenge lies in the fact that the investigated models either do not allow to reach sufficiently high default correlation levels (as it is the case when, among others, stochastic hazard rates are modeled as correlated geometric Brownian motions, see [Duffie and Garleanu (2001), Chapovsky et al. (2006)]) or become very quickly untractable from a practical point of view. Among the alternatives proposed in the literature is the Hull & White model [Hull and White (2008)]. In that paper, satisfactory correlation levels are got by introducing jumps in the hazard rates. More precisely, the name-specific hazard rates are decomposed as a sum of a deterministic continuous part and a pure-jump random process, so that a calibration of the continuous part must be performed in order for the model to be in-line with single-name probabilities. These are typically obtained from single-name market of Credit Default Swaps (CDS).

Defining the random default time of entity i by τ_i, the key point here is that the single-name survival probabilities $S_i(t) \doteq \Pr[\tau_i > t]$ are modeled in two different ways depending if we are considering them as

*Dr F. Vrins is with the Financial Markets/Credit Derivatives Modeling Team of ING SWE. Contact: +32 2 557 13 13, frederic.vrins@ing.be, Aue Marnix 24, M2+4, 1050 Brussels, Belgium. The author is grateful to John Hull for discussions on the present model.

isolated entities or as a part of a portfolio. From the CDS market, we infer the risk-neutral (deterministic, piece-wise constant) hazard rate curves $(\lambda_i(s))_{s\geqslant 0}$, so that :

$$S_i(t) = \mathrm{e}^{-\int_{s=0}^{t}\lambda_i(s)ds} \quad . \tag{1}$$

When focusing on the portfolio, the hazard becomes stochastic :

$$\hat{S}_i(t) \doteq \mathrm{e}^{-X_i(t)}, \qquad X_i(t) \doteq \int_{s=0}^{t}\mu_i(s)ds + \sum_{j=1}^{J_t}H(j) \ ,$$

where $(\mu_i(s))_{s\geqslant 0}$ is a continuous, deterministic function, $H(j) > 0$ is again a determinisitic function defined on the set of natural numbers and $(J_s)_{s\geqslant 0}$ is a non-homogeneous Poisson process with time-t probability mass function (see e.g. [Schoutens (2003)]):

$$\Pr[J_t = k] = \frac{\mathrm{e}^{-\Lambda(t)}(\Lambda(t))^k}{k!}, \qquad \Lambda(t) \doteq \int_{s=0}^{t}\lambda(s)ds \ .$$

The quantity $\lambda(t)$ is the time-t jump intensity of the process $(J_s)_{s\geqslant 0}$ (hence, $\lambda(s) \geqslant 0$ for all s). Recall that dJ_s is binary : either 1 or 0. Because the jump process is the same for all variables, this is the "common" variable creating the dependency. Conditionally on $J_t = j$, the defaults are independent[1]. As a corollary, the cases $\lambda(s) = 0 \ \forall s \leqslant t$ or $H = 0$ correspond to independence. In the sequel, we shall assume that $H(j) = H > 0$ is constant, so that $\sum_{j=1}^{J_t} H(j) = J_t H$. Following [Hull and White (2008)], we shall use the short-hand notation $J \doteq J_t = \int_{s=0}^{t} dJ_s$ when no ambiguity is possible about the time-value at which the Poisson process is evaluated.

The purpose if this paper is to complete the Hull & White model with analytical expressions for key quantities. This will avoid iterations for calibration of default probabilities. Further, the numerical integration of J-conditional expected number of defaults could be avoided in some specific cases with the help of the closed form expression for the survival distribution being derived below. These results could also be used to extend the application of the model to baskets with non-homogeneous recoveries as well. Throughout this paper, special emphasis is put on nth-to-default swaps.

The remaining of this note is organized as follows. First we show that there is a closed-form expression for the μ_i such that single-name calibration is ensured. It turns out that the survival probability up to time t is given by a simple rescaling of that obtained assuming independence. This scaling function, called the *jointure function*, is proved to play a key role in the Hull & White model; this is shown in Section 2. In Section 3, pairwise distributions and conditional distributions are worked out, and the probability to have simultaneous defaults is emphasized. This will be extended to the N-dimensional case. The distribution of the nth default time is then computed in Section 4. The first-to-default (FtD) case proves to have an extremely simple form. Finally, the explicit formula of correlation between default indicators, which we call *default correlation*, is given. Its is used to compare the evolution of the default correlation under the static Gaussian copula with respect to the dynamic models of Hull & White and Mai & Scherer; this is the purpose of Section 5. Finally we tackle the major problem of this class of pure-jump models, which is the possibility of facing simultaneous defaults at a time, preventing a reliable handling of cases where baskets are built of underlyings having different recovery rates. To that end, we propose in Section 6 a "first-order" approach towards proper dealing of the non-homogeneous case by splitting events where simultaneous defaults happen or not.

[1] One could question this as there is a remaining component in all hazard rate curves, $\Lambda(t)$. However, this component is deterministic, so that each conditional survival probability is determined by deterministic hazard rate curve, and hence they are indeed independent. See for example [Papageorgiou and Sircar (2008)] for more details.

2 Calibration of the μ_i

In order for our model survival probabilities to be in line with the values got from CDS market, it is needed that the expected value of the conditional survival probability $S_i(t|J) \doteq \hat{S}_i(t|J)$ matches $S_i(t)$:

$$S_i(t) = \mathbb{E}[S_i(t|J)] \ . \tag{2}$$

Hence, with $M_i(t) \doteq \int_{s=0}^{t} \mu_i(s)ds$, and $S_i(t|J) \doteq \mathrm{e}^{-M_i(t) - \sum_{j=1}^{J} H(j)}$, we have

$$\begin{aligned} S_i(t) &= \mathrm{e}^{-M_i(t)}\, \mathbb{E}\left[\mathrm{e}^{-JH}\right] \\ &= \mathrm{e}^{-M_i(t)}\, \phi_t(-H) \ , \end{aligned} \tag{3}$$

where $\phi_t(u) \doteq \mathbb{E}[\mathrm{e}^{uJ}]$ is the moment generating function of $J = J_t$. This function will be very helpful in the sequel. It admits the handy closed-form expression

$$\phi_t(u) = \mathrm{e}^{\Lambda(t)(\mathrm{e}^u - 1)} \ .$$

With this in hands, equating R.H.S. of Eq. 1 with that of Eq. 3 yields

$$\begin{aligned} M_i(t) &= \log(\phi_t(-H)) - \log(S_i(t)) \\ \mu_i(t) &= \frac{dM_i(t)}{dt} \\ &= (\mathrm{e}^{-H} - 1)\frac{d\Lambda(t)}{dt} - \frac{d\log(S_i(t))}{dt} \ . \end{aligned}$$

Using $\frac{d\Lambda(t)}{dt} = \lambda(t)$ and $\lambda_i(t) = f_i(t)/S_i(t)$ with $f_i(t) = -dS_i(t)/dt$, one gets the calibration equation

$$\mu_i(t) = \lambda(t)(\mathrm{e}^{-H} - 1) + \lambda_i(t) \quad \Leftrightarrow \quad \lambda_i(t) = \mu_i(t) - \lambda(t)(\mathrm{e}^{-H} - 1) \ . \tag{4}$$

The above expression leads to "auto-calibrated" equations. Indeed, with $\Lambda_i(t) \doteq \int_{s=0}^{t} \lambda_i(s)ds$, replacing $M_i(t)$ by $\Lambda(t)(\mathrm{e}^{-H} - 1) + \Lambda_i(t)$ will automatically ensure that Eq. 4 holds.

As a corollary, we have that

$$\mathbb{E}\left[\prod_{i=1}^{n} S_i(t|J)\right] = \mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{n}(M_i(t) + HJ)}\right] = \psi(n, H, \Lambda(t)) \prod_{i=1}^{n} S_i(t) \ , \tag{5}$$

where we have used

$$\begin{aligned} \psi(n, H, \Lambda(t)) &\doteq \phi_t(-nH)\, \mathrm{e}^{-n\Lambda(t)(\mathrm{e}^{-H} - 1)} \\ &= \mathrm{e}^{\Lambda(t)\left((\mathrm{e}^{-nH} - 1) - n(\mathrm{e}^{-H} - 1)\right)} \ . \end{aligned} \tag{6}$$

Note that the L.H.S. of Eq. 5 is nothing but the probability that the names $1, \ldots, n$ all survive up to time t, as given in the Hull & White model. Therefore, with $S^{\perp}(t) \doteq \prod_{i=1}^{n} S_i(t)$ to denote the joint survival probability up to time t under independence :

$$\begin{aligned} S(t) &= \Pr[\tau_i > t \ldots, \tau_n > t] \\ &= \psi(n, H, \Lambda(t)) \prod_{i=1}^{n} \Pr[\tau_i > t] \\ &= \psi(n, H, \Lambda(t)) S^{\perp}(t) \ . \end{aligned} \tag{7}$$

In the following, we shall make use of short-hand notation $\psi(n) \doteq \psi(n, H, \Lambda(t))$ and call ψ the *jointure function* for obvious reasons.[2]

Lemma 2.1 *[Porperties of the jointure function]*
The jointure function satisfies[3]

$$\psi(n, 0, \Lambda(t)) = \psi(n, H, 0) = 1$$

and

$$\frac{d\psi(n, H, \Lambda(t))}{dt} = \psi(n, H, \Lambda(t)) \log\Big(\psi(n, H, \lambda(t))\Big) \ .$$

Further, if $H > 0, \Lambda(t) > 0$, then $\psi(n, H, \Lambda(t)) > 1$ for all $n > 1$.

The proof of this lemma is skipped due to its simplicity (note : last result can be proven by induction).

3 Some results for the $N = 2$ case

It is informative to work out some results in the two-dimensional case; they will be useful for analyzing pairwise default correlation for instance. We first compute the joint distribution

$$F_{ij}(x, y) = \Pr[\tau_i \leqslant x, \tau_j \leqslant y] \ .$$

This will allow us to check that simultaneous defaults are possible in that model, that is that $\Pr[\tau_i = t|\tau_j = t] > 0$. Further, this is useful to compute correlation between the default indicators $A_i(t), A_j(t)$ where

$$A_i(t) \doteq \mathbb{I}_{\{\tau_i \leqslant t\}}$$

is the random variable indicating whether name i defaulted by time t or not.

3.1 Distribution

We suppose first that $y > x$ and that calibration of the $(\mu_i(s))_{0 \leqslant s \leqslant t}$ is achieved via Eq. 4.

$$\begin{aligned} F_{ij}(x, y) &= \mathbb{E}\left[\left(1 - \mathrm{e}^{-\int_0^x \mu_i(s)ds - J_x H}\right)\left(1 - \mathrm{e}^{-\int_0^y \mu_j(s)ds - J_y H}\right)\right] \\ &\overset{J_y = J_x + (J_y - J_x)}{=} 1 - S_i(x) - S_j(y) + \mathrm{e}^{-M_i(x) - M_j(y)}\, \mathbb{E}\left[\mathrm{e}^{-2J_x H - (J_y - J_x)H}\right] \ . \end{aligned}$$

Under the additional assumption that the jump intensity is constant over time, $\lambda(t) = \lambda$ (hence $\Lambda(t) = \lambda t$), J_t becomes a homogeneous Poisson process and it comes that

$$J_y - J_x \sim J_{y-x}, \quad J_{y-x} \perp J_x \ ,$$

where $X \perp Y$ has to be understood as (X, Y) being independent. Hence,

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-2J_x H - (J_y - J_x)H}\right] &= \mathbb{E}\left[\mathrm{e}^{-2HJ_x}\right] \mathbb{E}\left[\mathrm{e}^{-HJ_{y-x}}\right] \\ &= \phi_x(-2H)\phi_{y-x}(-H) \ . \end{aligned}$$

Consequently, extending this result to the case $x \leqslant y$ one gets

$$F_{ij}(x, y) = 1 - S_i(x) - S_j(y) + \mathrm{e}^{-M_i(x) - M_j(y)}\, \mathrm{e}^{\lambda(\mathrm{e}^{-H} - 1)((x \wedge y)(\mathrm{e}^{-H} + 1) + |y - x|)} \ .$$

[2] The above structure is close to that given by Sibuya [Sibuya (1960)], where the joint distribution $H(x, y)$ is modeled as the product of the marginals F (G) of X (Y) (evaluated at x and y respectively) times a function $\Omega(F(x), G(y))$ usually called *Sibuya* function. It is different, though, as i) here we are working with survival distributions instead of cumulative ones and ii) our jointure function is of the form $\Omega^\star(x, y)$ instead of $\Omega(F(x), G(y))$. Our *jointure function* precisely corresponds to the *local dependence index* as defined by Anderson et al., see e.g. [Anderson et al. (1992)], [Drouet Mari (1999)]. We are grateful to Prof. Jaap Spreeuw from Cass Business School (City University, London) for pointing that out.

[3] The $H = 0$ or $\Lambda(t) = 0$ cases corresponds to $\mathbb{E}\left[\prod_{i=1}^n S_i(t|J)\right] = \prod_{i=1}^n \mathbb{E}[S_i(t|J)]$, that is independence, as argued above

3.2 Conditional probability $\Pr[\tau_i \leqslant x|\tau_j = y]$

The conditional probability $F_{i|j}(x|y) \doteq \Pr[\tau_i \leqslant x|\tau_j = y]$ is obtained by computing $\frac{\partial F_{ij}(x,y)}{\partial y}/f_j(y)$ where $f_j(y) = -\frac{dS_j(y)}{dy}$ is the marginal density function of τ_j. Straightforward algebraic manipulations yield, together with the assumption $\lambda(t) = \lambda$,

$$\begin{aligned} F_{i|j}(x|y) &= S_j(y)\lambda_j(y) \\ &+(-\mu_j(y) + (\mathrm{e}^{-H} - 1)\lambda\chi(y))\,\mathrm{e}^{-M_i(x)-M_j(y)+\lambda(\mathrm{e}^{-H}-1)((x\wedge y)(\mathrm{e}^{-H}+1)+|y-x|)} \end{aligned}$$

with

$$\chi(y) \doteq \begin{cases} \mathrm{e}^{-H} & (y < x) \\ 1 & (y > x) \end{cases} .$$

Hence, there is a discontinuity in this CDF at $x = y$ provided that both $H > 0, \lambda > 0$, showing that there is a non-zero probability that $\tau_i = x$ given $\tau_j = x$ when there is a positive probability to have downwards jumps in the survival distribution. In other words, simultaneous defaults are indeed possible in the case τ_i and τ_j are not independent.

3.3 Positive quadrant dependency

It can be shown (see Lemma 6.2 below) that

$$S(t_1, t_2) = S_1(t_1)S_2(t_2)\psi(2, H, t_1 \wedge t_2) \ .$$

Because $\psi \geqslant 1$, the model exhibits *positive quadrant dependency* (see e.g. [Drouet Mari (1999)] or [Cherubini et al. (2006)]), and hence (Pearson's) correlation is always positive. Also, Bayes' rule implies

$$\Pr[\tau_1 \geqslant t_1|\tau_2 \geqslant t_2] \geqslant \Pr[\tau_1 \geqslant t_1] \ .$$

In other words, the information about the default time of name 2 arriving after some time $t_2 < \infty$ tends increase the probability of name 1 defaulting after t_1. This is an interesting feature as it mimics the impact of good news propagation : knowing that a company survives in the near future tends to increase the probability that another firm could also survive in a near future as well.

4 Distribution of n-th to default time

Let $\tau^{(1)} \doteq \operatorname{argmin}_j \tau_j$ be the random time of the first default. Equation Eq. 7 directly yields

$$\Pr[\tau^{(1)} > t] = S(t) = \psi(N) \prod_{i=1}^{N} S_i(t) \ . \tag{8}$$

This result is extremely appealing : under this model the joint survival distribution of N names $S(t, \ldots, t)$ up to time t is just the one obtained by independence (product of individual survival distributions) adjusted by a correction factor, only depending of the jump size H and of time via the deterministic function $\lambda(t)$ (observe that there is no assumption about the "time-independency" of $\lambda(s)$ here) .

This is useful to price FtD swaps when all the names have the same loss-given-default (i.e. notionals and recoveries are not name-specific) as in this case this information combined to payment calendar and discount rates is enough to value the swap; see e.g. [Laurent and Gregory (2003)].

The more general n-th to default case (NtD, $1 \leqslant n \leqslant N$, where N is the number of constituents in the basket) is also available in closed form and tractable for small N (as it is typically the case for NtD). First, we observe that if $\tau^{(1)} \leqslant \tau^{(2)} \leqslant \ldots \leqslant \tau^{(N)}$ is the ordered list of the default times, it holds

$$\Pr[\tau^{(n)} > t] \;=\; \sum_{j=0}^{n-1} \Pr\left[\sum_{i=1}^{N} A_i(t) = j\right] = \sum_{j=0}^{n-1} \Pr\left[N(t) = j\right]$$

with $N(t) \doteq \sum_{i=1}^{N} A_i(t)$ the number of defaults up to time t.

Taking into account all the possibilities that exactly n out of N names defaulted prior to t and exploiting conditional independence :

$$\begin{aligned}\Pr\left[N(t) = n|J\right] &= \sum_{\substack{1\leqslant i_1 <\cdot< i_n \leqslant N \\ \{i_1,\cdot,i_n,j_1,\cdot,j_{N-n}\}=\{1,\cdot,N\}}} \Pr[\tau_{i_1} \leqslant t, \cdot, \leqslant \tau_{i_n} \leqslant t, \tau_{j_1} > t, \cdot, \tau_{j_{N-n}} > t|J] \\ &= \sum_{\substack{1\leqslant i_1 <\cdot< i_n \leqslant N \\ \{i_1,\cdot,i_n,j_1,\cdot,j_{N-n}\}=\{1,\cdot,N\}}} \prod_{k=1}^{n} \Big(1 - S_{i_k}(t|J)\Big) \prod_{k'=1}^{N-n} S_{j_{k'}}(t|J) \;\; .\end{aligned}$$

The interpretation of the sum subscript above is that we partition the set $\{1, \ldots, n\}$ in two subsets of size n and $N - n$, respectively. The elements of the first set ($\{i_1, \ldots, i_n\}$) are assumed to have defaulted prior to t, not the others. The choice of the members of the first set uniquely determined those of the other set ($\{j_1, \ldots, j_{N-n}\}$). Finally, the condition $\{1 \leqslant i_1 < \cdots < i_n \leqslant N\}$ avoids counting twice the same events in terms of default indicators, as the permutation of elements in a same subset leads to the same partitioning. Note that we need to define

$$\prod_{k'=1}^{0} S_{j_{k'}}(t|J) \doteq 1 \tag{9}$$

to obtain the correct result in the case $n = N$. The expansion of the above product of default probabilities can be dealt with by applying the result

$$\prod_{i=1}^{n}(1 - \mathrm{e}^{x_i}) = 1 + \sum_{l=1}^{n}(-1)^l \sum_{1\leqslant j_1 <\ldots< j_l \leqslant n} \mathrm{e}^{-\sum_{z=1}^{l} x_{j_z}} \quad .$$

with $x_i \leftarrow -\Lambda_i(t) - JH$ (see [Mai and Scherer (2009)]). Hence,

$$\begin{aligned}\Pr\left[N(t) = n\right] &= \sum_{\substack{1\leqslant i_1 <\cdot< i_n \leqslant N \\ \{i_1,\cdot,i_n,j_1,\cdot,j_{N-n}\}=\{1,\cdot,N\}}} \mathbb{E}\left[\prod_{k'=1}^{N-n} S_{j_{k'}}(t|J)\right] \\ &\quad + \sum_{l=1}^{n}(-1)^l \sum_{1\leqslant m_1 <\cdot< m_l \leqslant n} \mathbb{E}\left[\prod_{z=1}^{l} S_{i_{m_z}}(t|J) \prod_{k'=1}^{N-n} S_{j_{k'}}(t|J)\right] \\ &\overset{Eq.\ 5}{=} \sum_{\substack{1\leqslant i_1 <\cdot< i_n \leqslant N \\ \{i_1,\cdot,i_n,j_1,\cdot,j_{N-n}\}=\{1,\cdot,N\}}} \prod_{k'=1}^{N-n} S_{j_{k'}}(t) \times \Big\{\psi(N-n) \\ &\quad + \sum_{l=1}^{n}(-1)^l \sum_{1\leqslant m_1 <\cdot< m_l \leqslant n} \psi(N+l-n) \prod_{z=1}^{l} S_{i_{m_z}}(t)\Big\} \;\; .\end{aligned} \tag{10}$$

Observe that calibration to CDS probabilities is already included in these equations.

5 Correlation of default indicators

In this section, we first derive the pairwise default correlations, defined as $\mathrm{Corr}(A_i(t), A_j(t))$, the correlation between default indicators, in the Hull & White model. We then compare the shapes under the three various models mentioned in the introduction.

5.1 Default correlations under the Hull & White, the Gaussian copula and the Mai-Scherer models

The covariance between indicator functions is :

$$\begin{aligned}
\mathrm{Cov}\left(\mathbb{1}_{\{\tau_i\leqslant t\}}, \mathbb{1}_{\{\tau_j\leqslant t\}}\right) &= \mathbb{E}[\mathbb{1}_{\{\tau_i\leqslant t\}}\,\mathbb{1}_{\{\tau_j\leqslant t\}}] - \mathbb{E}[\mathbb{1}_{\{\tau_i\leqslant t\}}]\,\mathbb{E}[\mathbb{1}_{\{\tau_i\leqslant t\}}] \\
&= F_{ij}(t,t) - F_i(t)F_j(t) \\
&= \Big(1 - S_i(t) - S_j(t) + S_{ij}(t,t)\Big) - (1 - S_i(t))(1 - S_j(t)) \\
&= S_{ij}(t,t) - S_i(t)S_j(t) \ .
\end{aligned}$$

The variance of $\mathbb{1}_{\{\tau_i\leqslant t\}}$ is given by

$$\mathbb{E}[\mathbb{1}^2_{\{\tau_i\leqslant t\}}] - \mathbb{E}^2[\mathbb{1}_{\{\tau_i\leqslant t\}}] = F_i(t) - F_i^2(t) = F_i(t)S_i(t) \ .$$

So that,

$$\mathrm{Corr}(A_i(t), A_j(t)) = \frac{F_{ij}(t,t) - F_i(t)F_j(t)}{\sqrt{F_i(t)F_j(t)}\sqrt{S_i(t)S_j(t)}} = \frac{S_{ij}(t,t) - S_i(t)S_j(t)}{\sqrt{F_i(t)F_j(t)}\sqrt{S_i(t)S_j(t)}} \ . \tag{11}$$

5.1.1 Hull & White

For the Hull & White model, Eq. 7 gives $S_{ij}(t,t) = \psi(2)S_i(t)S_j(t)$ and the correlation is given by the simple expression

$$\mathrm{Corr}(A_i(t), A_j(t)) = (\psi(2) - 1)\sqrt{\frac{S_i(t)S_j(t)}{F_i(t)F_j(t)}} \ . \tag{12}$$

We observe that at first sight, the Hull & White model can potentially leads to correlation higher than 1 when the default probability are taken sufficiently small and for high jump size H occurring with high likelihood ($\lambda(s)$ high). However, this cannot happen if we constraint $M_i(t) > 0$. This is proven in the appendix.

5.1.2 Gaussian Copula

The default probability is here given by the bivariate normal distribution [Li (1999)] :

$$F_{ij}(t,t) = \Phi_2\Big(\Phi^{[-1]} \circ F_i(t), \Phi^{[-1]} \circ F_j(t), \sqrt{\rho}\Big) \ ,$$

so that,

$$\mathrm{Corr}(A_i(t), A_j(t)) = \frac{\Phi_2\Big(\Phi^{[-1]} \circ F_i(t), \Phi^{[-1]} \circ F_j(t), \sqrt{\rho}\Big) - F_i(t)F_j(t)}{\sqrt{F_i(t)F_j(t)}\sqrt{S_i(t)S_j(t)}} \ . \tag{13}$$

5.1.3 Mai and Scherer

This model assumes that i defaulted prior to time t if a stochastic process $\zeta_{\lambda_i(t)}$ reaches a random threshold E_i [Mai and Scherer (2009)] :

$$A_i(t) = 1\!\!1_{\{\tau_i \leqslant t\}} = 1\!\!1_{\{\zeta_{\lambda_i(t)} \geqslant E_i\}} \quad .$$

More precisely, $E_i \sim \exp(1)$, $i \in \{1, \ldots, n\}$ are i.i.d. random default thresholds, and ζ_t is a Lévy subordinator[4] with Laplace exponent Ψ independent of the E_i's : $\mathbb{E}[\mathrm{e}^{s\zeta_t}] = \mathrm{e}^{t\Psi(s)}$ (Recall that the Laplace exponent $\Psi(x)$ is nothing but the *Characteristic exponent* of the process $(X_s)_{s\geqslant 0}$ evaluated at $-ix$: $\Psi(x) = \tilde{\Psi}(-ix)$, $\tilde{\Psi}(x) \doteq \log(\mathbb{E}[\mathrm{e}^{iuX_1}])$, see e.g. [Cont and Tankov (2004)])[5]. When $\zeta_t \doteq \mu t + \sum_{l=1}^{N_t} G_l$ where N_t is a Poisson process with intensity β and G_l are i.i.d. $\exp(\eta)$ random variables, ζ_t is a compound Poisson process. In this case, we have

$$\Psi(s) = \mu s + \beta s/(\eta - s) \quad . \tag{14}$$

The calibration with respect to default probabilities is achieved via the constraint $\Psi(-1) = -1$, which uniquely determines μ as being equal to $1 + \beta/(1 + \eta)$ (this is the equivalent of Eq. 4 in the Hull & White model). In the sequel, we fix $\beta = 1$, which allows η to be chosen arbitrarily in $\mathbb{R}^+$. Finally, the default correlation is given by

$$\begin{aligned}\mathrm{Corr}(A_i(t), A_j(t)) \quad &= \quad \frac{\Big(1 - (F_i(t) \wedge F_j(t))^{-(\Psi(-2)+1)}\Big)\Big(1 - (F_i(t) \vee F_j(t))\Big)}{\sqrt{F_i(t)F_j(t)S_i(t)S_j(t)}} \\ &\quad + \frac{F_i(t) + F_j(t) - 1 - F_i(t)F_j(t)}{\sqrt{F_i(t)F_j(t)S_i(t)S_j(t)}}\end{aligned} \tag{15}$$

and the survival probability by

$$\Pr[\tau_i > t, \tau_j > t] = S_{ij}(t,t) = (S_i(t) \wedge S_j(t))(S_i(t) \vee S_j(t))^{\Psi(-1)-\Psi(-2)} \quad .$$

5.2 Default correlation comparison

Very often, models are compared based on market-implied quantities. For instance, we compare time-stability of market-implied parameters or how good is the fit of models when applied to multiple tranches. We believe this equation is also useful for model comparison when we are lacking market data, as it is e.g. the case regarding NtD swaps. For instance, suppose that we don't know the price of the FtD because of a lack of liquidity. But suppose that analysts are confident in the survival probability by time T. In such a case, it makes sense to compare models under the constraint that they have the same survival probability up to T. For instance, we could compare correlation of default indicators coming from different models under constraint that they have the same time-T survival probability. In the sequel, we shall do this exercise to compare the Gaussian Copula, the Hull & White and the Mai-Scherer model with compound Poisson process chosen as Lévy subordinator, that is with Laplace exponent given by Eq. 14.

5.2.1 Two-dimensional case

In the simulation below, we fix $H, \lambda(t) = \lambda, \lambda_i(t) = \lambda_i$ and $\lambda_j(t) = \lambda_j$ with $T = 5$. We compute the survival probability by time T, $\Pr[\tau_i > t, \tau_j > t]$, as given by the Hull & White model using $\psi(2)\,\mathrm{e}^{-t(\lambda_i+\lambda_j)}$, and calibrate the loading factor ρ of the Gaussian copula and the rate parameter η of the Mai & Scherer model to obtain the same probability $S(T)$, that is we look for ρ and η such that the $S(T)$ are equal for all three models.[6]

[4] Recall that a Lévy subordinator is nothing but a non-decreasing Lévy process.

[5] This directly results from Lévy processes theory : $\mathbb{E}[\mathrm{e}^{s\zeta_t}] = (\Phi_{\zeta_1}(-is))^t = (\mathrm{e}^{\Psi(s)})^t = \mathrm{e}^{t\Psi(s)}$ since $\Psi(s) \doteq \log(\Phi_{\zeta_1}(-is))$ is the log of the ζ_1-characteristic function evaluated at $(-is)$.

[6] In the two-dimensional case, calibrating on $F_{ij}(t,t)$ or $S_{ij}(t,t)$ is equivalent.

Model	Margins		H & W		GC	M & S	
Case	$\lambda_i(t)$	$\lambda_j(t)$	H	$\lambda(t)$	ρ	β	η
1	0.05	0.05	5	0.01	57%	1	1.68
2	0.05	0.02	0.01	0.82	16 %	1	9
3	0.001	0.001	10	0.001	99.5 %	1	0.05%

Table 1: Testing cases for model comparison leading to a same survival probability $S_{ij}(t) \doteq \Pr[\tau_i > t, \tau_j > t]$ at maturity $t = T$ together with the implied loading factor of the Gaussian copula ρ. We also give the implied rate η for the Mai-Scherer model when a compound Poisson process with rate 1 and threshold distributed as $\exp(\eta)$ is chosen as the Lévy subordinator.

Equipped with this calibration procedure, we can now have a look at the evolution of the survival probability as a funciton of t and at the correlation $\mathrm{Corr}(A_i(t), A_j(t))$ with respect to time for the three models. This exercise is performed on the three various test cases described in Table 1 and is illustrated in Figure 1.

We note $S^{HW}(t)$, $S^{GC}(t)$ and $S^{MS}(t)$ the survival probabilities up to time t, that is the generated $S_{ij}(t,t)$ for each model and define $Err(t) = S^{HW}(t) - S^{GC}(t)$ and $Err(t) = S^{HW}(t) - S^{MS}(t)$. The first interesting phenomenon to observe is that the survival probability curves are very similar. This means that calibration of the curves at maturity gives a pretty good fit for all $t \leqslant T$ as well. This tends to indicate that all models behave similarly with respect to time. Another quantity of interest is the default correlation. One can see that contrarily to the Gaussian copula, both jump-based models tend to produce a default correlation being much more constant over time. Further, in the Gaussian case, the short-term default correlation tends to zero when so does t; for very short maturities, the correlation between default indicators tends to zero. This is not the case for the Hull & White model with the proposed calibration scheme. Same holds true for the Mai & Scherer model. Finally, another key difference is that the Hull & White model allows to reach much higher levels of default correlation; this is emphasized in Case 3, where a value of $\rho = 99.5\%$ only leads to a default correlation less than 85%. By contrast, the Hull & White model with the same survival probability at maturity allows to have default correlations up to 100%. Regarding the Mai & Scherer model, even if the absolute levels of default correlation do not exactly match those of the Hull & White models, they are shown to be rather close, and exhibit the same slowly decreasing behaviour as time passes.

5.2.2 N-dimensional case

As previously stated, the calibration of the model parameters in order to obtain a same survival probability at maturity $\Pr[\tau^{(1)} > T]$ for each model will lead to the same prices for FtD swaps with basket built of two constituents. This is because of the joint effect that i) FtD prices are more dependent upon the presence of a default or not before maturity than to the exact defaulting time, and ii) the implied survival probability curves are very similar for $0 < t < T$. It is instructive, however, to have a look at "last to default"-related quantities coming out of the three models under the constraint that they all lead to a same value of the survival probability $S(T) = \Pr[\tau^{(1)} > T]$ for larger baskets. Note that this is just the survival probability evaluated at $(t_1, \ldots, t_n) = (T, \ldots, T)$. This quantity could not be easily obtained for the Gaussian copula as the last model rather provides the distribution function. However, the above no-default probability can be readily obtained via the integral

$$S^{GC}(T) = \int_{z=-\infty}^{\infty} \prod_{i=1}^{N} \left(1 - \Phi\left(\frac{\Phi^{[-1]}(F_i(t)) - \rho Z}{\sqrt{1-\rho^2}}\right)\right) d\Phi(z)$$

which could be efficiently evaluated with the help of some quadrature.

These probability curves can be obtained as follows, for the One factor Gaussian Copula, Hull & White and Mai & Scherer models :

$$\begin{aligned}
\Pr[N^{GC}(t)=N] &= \int_{z=-\infty}^{\infty}\prod_{i=1}^{N}\Phi\left(\frac{\Phi^{[-1]}(F_i(t))-\rho z}{\sqrt{1-\rho^2}}\right)d\Phi(z)\\
\Pr[N^{HW}(t)=N] &\overset{Eq.\ 9\ \&\ 10}{=} 1+\sum_{l=1}^{N}(-1)^l\psi(l)\sum_{1\leqslant j_1<\ldots<j_l\leqslant N}\prod_{z=1}^{l}S_{j_z}(t)\\
\Pr[N^{MS}(t)=N] &= 1+\sum_{l=1}^{N}(-1)^l\sum_{1\leqslant j_1<\ldots<j_l\leqslant N}C_l^{\nu}(S_{j_1}(t),\ldots,S_{j_l}(t))
\end{aligned}$$

where we have used

$$(x_1,\ldots,x_l)\mapsto C_l^{\nu}(x_1,\ldots,x_l)=x_{(1)}\prod_{i=2}^{l}x_{(i)}^{\Psi(-(i-1))-\Psi(-i)}$$

with $x_{(1)}\leqslant x_{(2)}\leqslant\ldots\leqslant x_{(N)}$ is the ordered list of $\{x_1,\ldots,x_N\}$.

One can see on Figure 2 that the impact for that "tranche" may be far from being negligible. We assumed a 5-names basket (N=5) with default intensity vector $(5.17\%, 8.2\%, 6.87\%, 5.4\%, 9.7\%)$. We have chosen $(H,\lambda)=(10,1\%)$ yielding $\Pr[N(T)=0]=20.9\%$. The implied values for the two other models are 44.5% (Gaussian correlation) and $(\beta,\eta)=(1,2.86)$. Survival probability curves are shown in Panel 2(a) and converges to the same value at maturity, which shows that proper calibration is achieved. Panel 2(b) shows the difference between Hull & White curve compared to the others, $Err(t)=S^{HW}(t)-S^{\cdot}(t)$. Pairwise default correlations are shown in Panel 2(c). Finally, one can see in Panel 2(d) that the model-implied curves $\Pr[\tau^{(N)}<T]$ are, in spite of the calibration, very different. Both jump-based models rend extreme scenario much more likely than the Gaussian copula one. The Hull & White model gives the maximum probability for this "catastrophe" scenario to happen.

6 A route for pricing non-homogeneous recovery FtD baskets

The major drawback of jump-based models like that of Mai & Scherer and of Hull & White for instance, is that they do not allow to price baskets which have name-specific recoveries. Indeed, in this case, the contingent leg of a n-th to default basket is given in the literature by

$$\int_{t=0}^{T}\delta(t)\sum_{i=1}^{N}(1-R_i)\lim_{ds\to 0}\frac{\Pr[\tau_i\in[t,t+ds),\sum_{j\neq i}A_j(t)=n-1]}{ds}dt \tag{16}$$

where $\delta(t)$ is the discount factor, N is the number of constituents in the basket, R_i is the recovery rate of entity i ($i\in\{1,\ldots,N\}$), T is the maturity of the deal and the above probability is that of the i-th name causing the n-th default in the interval $[t,t+ds)$. There is a major assumption encapsulated in the above formula : it only allows one name to default at a time. This requirement is met for continuous copula models, like the Gaussian copula (see for example [Laurent and Gregory (2003)]). This is not true in the present case, however, as simultaneous are proven to be possible (see Section 3.2). This becomes non-negligible as the size of the jumps or the rate of them become moderately high.

In order to see this, consider the following lemma (for which a proof is given in the Appendix) :

Lemma 6.1 (Densities of first default and conditional density of τ_i given J) *The density of the first default time $\tau^{(1)}$ is given by*

$$f^{(1)}(t)=\left(\sum_{i=1}^{N}\lambda_i(t)-\log(\psi(N,H,\lambda(t)))\right)\psi(N,H,\Lambda(t))S^{\perp}(t)=\left(\sum_{i=1}^{N}\lambda_i(t)-\log(\psi(N,H,\lambda(t)))\right)S(t)$$

If further the intensity of the jumps is constant ($\lambda(t) = \lambda$), the density of τ_i conditional on $J = j$ is given by

$$f_i(t|j) = \lambda_i(t) S_i(t)\, \mathrm{e}^{-\Lambda(t)(\mathrm{e}^{-H} - 1) - jH}$$

Equation Eq. 16 would hold if the following is true

$$f^{(1)}(t) \overset{?}{=} \sum_{i=1}^{N} \lim_{ds \to 0} \frac{\Pr[\tau_i \in [t, t+ds), \sum_{j \neq i} A_j(t) = 0]}{ds} \tag{17}$$

Indeed, also in the specific case $n = 1$, we need Eq. 16 to be in-line with the homogeneous expression when $R_i = R$, which is

$$CL(R) = (1 - R) \int_{t=0}^{T} \delta(t) f^{(1)}(t) dt \tag{18}$$

We now compare the L.H.S. of Eq. 17 with that of the R.H.S. with the help of Lemma 6.1. It comes that using conditional independence, the R.H.S is

$$\begin{aligned}
\sum_{i=1}^{N} \mathbb{E}\left[\lim_{ds \to 0} \frac{\Pr\left[\tau_i \in [t, t+ds), \sum_{j \neq i} A_j(t) = 0 | J\right]}{ds}\right] &\overset{\perp}{=} \sum_{i=1}^{N} \mathbb{E}\left[f_i(t|J)\, \mathrm{e}^{-\sum_{j \neq i} M_j(t) - (N-1)JH}\right] \\
&= \sum_{i=1}^{N} \mathbb{E}\left[f_i(t|J)\, \mathrm{e}^{-\sum_{j \neq i} M_j(t) - (N-1)JH}\right] \\
&= S^{\perp}(t) \sum_{i=1}^{N} \lambda_i(t)\, \mathrm{e}^{-N\Lambda(t)(\mathrm{e}^{-H} - 1)}\, \mathbb{E}\left[\mathrm{e}^{-NJH}\right] \\
&= S^{\perp}(t) \sum_{i=1}^{N} \lambda_i(t)\, \mathrm{e}^{-N\Lambda(t)(\mathrm{e}^{-H} - 1)}\, \phi_t(-NH) \\
&= \left(\sum_{i=1}^{N} \lambda_i(t)\right) S^{\perp}(t) \psi(N) \\
&= \left(\sum_{i=1}^{N} \lambda_i(t)\right) S(t)
\end{aligned}$$

This shows that Eq. 17 does not hold when $\psi(N, H, \lambda) \neq 1$, which precisely the case for which independence is not achieved provided that $N > 1$ (this condition is assumed to be met as we are dealing with baskets including more than one single entity). The impact of multiple defaults can be evaluated by checking how far is the following ratio to 1:

$$\frac{\sum_{i=1}^{N} \lambda_i(t) - \log(\psi(N, H, \lambda))}{\sum_{i=1}^{N} \lambda_i(t)} \ .$$

Properties of jointure function show that this impact decreases as size of the jump decreases or as rate of jump arrivals decreases. This is very intuitive as then the impact of jumps decreased, and we converge to the independent case, given by the product copula.

It is possible, however, to try to go one step further in the context of the Hull & White model. In order to cope with that, we propose to split the contingent leg in two parts. The first one deals with the events where only one default is allowed at a time, and the second part deals with cases where multiple simultaneous defaults are allowed. In the first part, there will be no ambiguity about which recovery applies, and a formula similar to Eq. 16 could be used to value this part. In the second case, however, there will be an ambiguity, and some assumptions need to be made. In that case, one could decide to take the average of recovery rates, or to adopt a more conservative approach, for example taking the smallest or largest recovery (depending on the position we have); we note the chosen value for this recovery rate upon multiple defaults as $\hat{R}$.

The remaining is based on the two following lemmas, proven in the Appendix:

Lemma 6.2 (Survival distribution in n dimensions) *Under the assumption of homogeneous jumps, that is $\lambda(t) = \lambda$, the survival probability $\Pr[\tau_1 > t_1, \ldots, \tau_N > t_N]$ under the Hull & White model with constant jump size is given by*

$$S(t_1, \ldots, t_N) = \prod_{i=1}^{N} S_i(t_i)\psi\Big((N-i+1), H, \lambda(t_{(i)} - t_{(i-1)})\Big)$$

where $0 \doteq t_{(0)} \leqslant t_{(1)} \leqslant t_{(2)} \leqslant \ldots \leqslant t_{(N)}$ is the ordered version of $\{t_1, \ldots, t_N\}$.

In the case $t_1 = \ldots = t_N = t$, the jointure function is $\psi\Big(N, H, \lambda t\Big)$ (if $i = 1$) or 1 (otherwise) and therefore:

$$S(t, \ldots, t) = \psi\Big(N, H, \lambda t\Big) \prod_{i=1}^{N} S_i(t)$$

where one recognizes Eq. 5.

Lemma 6.3 (Densities of i-th default exclusively) *Under the assumption of homogeneous jumps, the probability that name i defaults in $[t, t+dt)$ and that all the remaining names survive up to $t+dt$ is given by*

$$S(t{+}dt, \ldots, t{+}dt, \underbrace{t}_{i}, t{+}dt, \ldots, t{+}dt) - S(t{+}dt) = S(t)\, \mathrm{e}^{-\sum_{j\neq i}\lambda_j(t)dt} \Big\{\psi(N-1, H, \lambda dt) - \psi(N, H, \lambda dt)\, \mathrm{e}^{-\lambda_i(t)dt}\Big\}$$

so that the associated density obtained by taking the limit of the ratio of the above difference by dt is given by

$$f_i^\star(t) \doteq \lim_{dt\downarrow 0} \frac{S(t+dt, \ldots, t+dt, t, t+dt, \ldots, t+dt) - S(t+dt)}{dt} = S(t)\left(\lambda_i(t) + \log\left(\frac{\psi(N-1, H, \lambda)}{\psi(N, H, \lambda)}\right)\right) \ .$$

The requirement for not to have multiple names defaulting at the same of the first one to default mathematically translates to $dN(\tau^{(1)}) = 1$: at the time of the first default, the total number of defaults increases by one. As a corollary,

$$\Pr[\tau^{(1)} \leqslant T, dN(\tau^{(1)}) = 1] = \sum_{i=1}^{N} \int_{t=0}^{T} f_i^\star(t)dt \ .$$

Consequently, one could evaluate the probability to have one default before time T which happened simultaneousley that at least one other obligor. Indeed, noting that

$$\Pr\Big[\tau^{(1)} \leqslant T\Big] = \Pr\Big[\tau^{(1)} \leqslant T, dN(\tau^{(1)}) = 1\Big] + \Pr\Big[\tau^{(1)} \leqslant T, dN(\tau^{(1)}) > 1\Big] \ ,$$

and combining this with Lemma 6.1 yields

$$\Pr\Big[\tau^{(1)} \leqslant T, dN(\tau^{(1)}) > 1\Big] = \int_{t=0}^{T} \left(f^{(1)}(t) - \sum_{i=1}^{N} f_i^\star(t)\right) dt \ .$$

(H,λ)	$\Pr[\tau^{(1)} \leqslant T]$	$\Pr[\tau^{(1)} \leqslant T, \Omega]$	$\Pr[\tau^{(1)} \leqslant T, \bar{\Omega}]$	$\log(\psi(N,H,\lambda))/\left(\sum_{i=1}^{N}\lambda_i(t)\right)$
(0,1%)	22.12	22.12	0	0
(10,0.1%)	20.55	20.1	0.45	8
(10,1%)	4.878	0.001	4.877	80

Table 2: Basket of $N = 5$ names with idiosyncratic rates $\lambda_i = 1\%$; values are expressed in [%]. The shorthand notations $\Omega \doteq \{dN(\tau^{(1)}) = 1\}$ and $\bar{\Omega} \doteq \{dN(\tau^{(1)}) > 1]\}$ have been used.

Based on these results, and supposing $\lambda_i(t) = \lambda_i$[7], we can turn the "homogeneous" contingent leg given in Eq.18 into

$$\begin{aligned}
CL(R_i) &= \sum_{i=1}^{N}(1-R_i)\int_{t=0}^{T}\delta(t)f_i^{\star}(t)dt + (1-\hat{R})\sum_{i=1}^{N}\int_{t=0}^{T}\delta(t)\Big(f^{(1)}(t) - f_i^{\star}(t)\Big)dt \\
&= \sum_{i=1}^{N}(1-R_i)\left(\lambda_i + \log\left(\frac{\psi(N-1,H,\lambda)}{\psi(N,H,\lambda)}\right)\right)\int_{t=0}^{T}\delta(t)S(t)dt \\
&\quad +(1-\hat{R})\Big((N-1)\log(\psi(N,H,\lambda)) - N\log(\psi(N-1,H,\lambda))\Big)\int_{t=0}^{T}\delta(t)S(t)dt \\
&= \Big\{\sum_{i=1}^{N}(1-R_i)\left(\lambda_i + \log\left(\frac{\psi(N-1,H,\lambda)}{\psi(N,H,\lambda)}\right)\right) \\
&\quad +(1-\hat{R})\Big((N-1)\log(\psi(N,H,\lambda)) - N\log(\psi(N-1,H,\lambda))\Big)\Big\} \times \int_{t=0}^{T}\delta(t)S(t)dt \qquad (19)
\end{aligned}$$

Obviously, the premium leg of the first-to-default does not depend on the recoveries, so it is exclusively dependent upon the rates (or discount factors) and the density of $\tau^{(1)}$, i.e. $f^{(1)}(t)$.

Observe that in the case of homogeneous recoveries, $R_i = R$, Eq. 19 will be in line with Eq. 18 if and only if $\hat{R} = R$, which is obviously the case if $\hat{R}$ is defined using percentile-based approach on the vector $(R_1, \ldots, R_N)$.

The above development shows that one could not use formula given by Eq. 16 to evaluate the default probability of a first to default swaps. The error made by using this expression vanishes when $H = 0$ or $\Lambda(t) = 0$, but increases as we deviate from this "independent" situation. This is illustrated in Table 2.

7 Conclusion

In this note, we have worked out the Hull & White dynamic model for pricing structured credit derivative [Hull and White (2008)]. It is shown that in the constant-jump size case $H(j) = H$, it turns out that calibration to underlying single names is automatically achieved thanks to closed-form expressions, just as for the barrier levels in the Gaussian copula case. Further, it is shown that the distribution of the n-th default time is also available in closed form, which leads to an extremely easy pricing of homogeneous first-to-default swaps, and remains tractable for typical $N > 1$. Correlation between default indicators are available analytically as well, which allows to derive some procedure in order to properly compare outcomes of various models. Compared to the Gaussian copula, for instance, the Hull & White model is proven to be able to yield higher default correlation levels that are non-vanishing as $t \to 0$. Survival distribution is also derived analytically, which allows model comparison of basket default swaps under calibration of the survival probability of the first default time at some point in time. This reveals another specificity of the model, which is to lead

[7]This is assumption is just made to allow us factorizing $\lambda_i(t) = \lambda_i$ in front of the integral, but the more general case where the name-specific hazard rate curves are time-dependent can be dealt with easily by leaving it inside the integral.

to a higher likelihood of extreme scenario even when calibration is performed following the aforementioned procedure. This is shared by the other jump-based model, and is a fundamental difference compared to the Gaussian copula. Finally, we dropped the restrictive constraint of the homogeneity of the recovery rates (typically needed for models allowing for simultaneous defaults) by developing a "first-order approach", in which simultaneous default cases are treated separately than those where the default is isolated.

8 Appendix

8.1 Default correlation range in Hull & White model

First, we observe that $\psi(2) = \mathrm{e}^{\Lambda(t)(\mathrm{e}^{-H}-1)^2} > 1$. This shows that the correlation between the default indicators up to t is positive (the model cannot generate negative correlations). In order to show that correlation is lower than one, we first work out the case $\lambda_i(s) = \lambda_j(s)$. Then, correlation $\rho_{ij} < 1$ reduces to $\psi(2)S_i < 1$, and the last inequality is proven when $\log(\psi(2)) < \Lambda_i(t)$. The last equality is readily seen to hold :

$$\begin{aligned}
\log(\psi(2)) &= \Lambda(t)(\mathrm{e}^{-H}-1)\,\mathrm{e}^{-H} - \Lambda(t)(\mathrm{e}^{-H}-1) \\
&\overset{Eq.\ 4}{=} \Lambda(t)(\mathrm{e}^{-H}-1)\,\mathrm{e}^{-H} - (M_i(t) - \Lambda_i(t)) \\
\log(\psi(2)) - \Lambda_i(t) &= \underbrace{\Lambda(t)}_{>0}\underbrace{(\mathrm{e}^{-H}-1)}_{<0}\underbrace{\mathrm{e}^{-H}}_{>0} - \underbrace{M_i(t)}_{>0} \\
&< 0
\end{aligned}$$

As a corollary we have that $(\psi(2)-1)\frac{S_i}{F_i} < 1$. Extending this to the general case $\Lambda_i(t), \Lambda_j(t)$, we first note that it is sufficient to show that $\Big((\psi(2)-1)S_i/F_i\Big)\Big((\psi(2)-1)S_j/F_j\Big) < 1$. Because each factor is lower than one as shown above, this concludes the proof.

8.2 Proof of Lemma 6.1

In order to prove the first result, we note that $\Pr[\tau^{(1)} > t] = S(t) = \psi(N, H, \Lambda(t))S^{\perp}(t)$, so that the density is given by

$$\begin{aligned}
f^{(1)}(t) &= -\frac{d\Big(\psi(N,H,\Lambda(t))\Big)S^{\perp}(t)}{dt} \\
&= -\frac{d\psi(N,H,\Lambda(t))}{dt}S^{\perp}(t) - \psi(N,H,\Lambda(t))\frac{dS^{\perp}(t)}{dt}
\end{aligned}$$

and the result comes by noting that

$$\frac{d\psi(N,H,\Lambda(t))}{dt} = \psi(N,H,\Lambda(t))\log\Big(\psi(N,H,\lambda(t))\Big)\ . \tag{20}$$

The second result is easy to prove too when the Poisson process J_t is homogeneous. In this case, the survival probability of name i up to time $t + dt$ is, for infinitesimal $dt \downarrow 0$

$$\begin{aligned}
S_i(t+dt|J) &= \mathrm{e}^{-M_i(t)}\,\mathrm{e}^{-\mu_i(t)dt}\,\mathrm{e}^{-J_tH}\,\mathbb{E}[\mathrm{e}^{-(J_{t+dt}-J_t)H}\,|J_t] \\
&= S_i(t|J)\,\mathrm{e}^{-\mu_i(t)dt}\,\phi_{dt}(-H) \\
&= S_i(t|J)\,\mathrm{e}^{-\lambda_i(t)dt}\,\mathrm{e}^{-\lambda(t)dt(\mathrm{e}^{-H}-1)}\,\mathrm{e}^{\lambda(t)dt(\mathrm{e}^{-H}-1)} \\
&= S_i(t|J)\,\mathrm{e}^{-\lambda_i(t)dt}\ ,
\end{aligned}$$

so that

$$\Pr[\tau_i \in [t, t+dt)|J] = S_i(t|J) - S_i(t+dt|J) = S_i(t|J)(1 - \mathrm{e}^{-\lambda_i(t)dt}) \ .$$

Using the Taylor expansion of the exponential

$$\mathrm{e}^{u.dt} = 1 + udt + u^2/2dt^2 + o(dt^2) \ , \tag{21}$$

the conditional density is given by

$$\begin{aligned} f_i(t|J) &= \lim_{dt\to 0} \frac{\Pr[\tau_i \in [t, t+dt)|J]}{dt} \\ &= S_i(t|J) \lim_{dt\to 0} \frac{1 - \mathrm{e}^{-\lambda_i(t)dt}}{dt} \\ &= S_i(t|J)\lambda_i(t) \ , \end{aligned}$$

and the claim results from Eq. 4.

8.3 Proof of Lemma 6.2

In order to compute the survival probability function for the Hull & White model, observe first that defining $t_{(0)} \doteq 0$

$$\sum_{i=1}^{N} t_i = \sum_{i=1}^{N} t_{(i)} = N(t_{(1)} - t_{(0)}) + (N-1)(t_{(2)} - t_{(1)}) + \ldots + (t_{(N)} - t_{(N-1)}) = \sum_{i=1}^{N}(N-i+1)(t_{(i)} - t_{(i-1)})$$

and using $J_0 = 0$

$$\sum_{i=1}^{N} J_{t_i} = \sum_{i=1}^{N} J_{t_{(i)}} = \sum_{i=1}^{N}(N-i+1)(J_{t_{(i)}} - J_{t_{(i-1)}}) \ .$$

Therefore, using independence of increments of homogeneous Poisson process and calibration equation Eq. 4.,

$$\begin{aligned} S(t_1, \ldots, t_N) &= \mathbb{E}\left[\prod_{i=1}^{N} \mathrm{e}^{-M_i(t_i) - J_{t_i} H}\right] \\ &= \mathrm{e}^{-\sum_{i=1}^{N} M_i(t_i)}\, \mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{N} J_{t_i} H}\right] \\ &\overset{\Lambda(t)=\lambda t}{=} \prod_{i=1}^{N} S_i(t_i)\, \mathrm{e}^{-\lambda(\mathrm{e}^{-H}-1)\sum_{i=1}^{N} t_i}\, \mathbb{E}\left[\mathrm{e}^{-H\sum_{i=1}^{N}(N-i+1)(J_{t_{(i)}} - J_{t_{(i-1)}})}\right] \\ &\overset{\perp}{=} \prod_{i=1}^{N} S_i(t_i)\, \mathrm{e}^{-\lambda(\mathrm{e}^{-H}-1)\sum_{i=1}^{N} t_i} \prod_{i=1}^{N} \mathbb{E}\left[\mathrm{e}^{-H(N-i+1)(J_{t_{(i)}} - J_{t_{(i-1)}})}\right] \\ &= \prod_{i=1}^{N} S_i(t_i)\, \mathrm{e}^{-\lambda(\mathrm{e}^{-H}-1)(N-i+1)(t_{(i)} - t_{(i-1)})}\, \phi_{t_{(i)} - t_{(i-1)}}(-(N-i+1)H) \\ &= \prod_{i=1}^{N} S_i(t_i)\psi\Big((N-i+1), H, \lambda.(t_{(i)} - t_{(i-1)})\Big) \ . \end{aligned}$$

8.4 Proof of Lemma 6.3

We start from the survival probability given in Lemma 6.2, with $t_i = t$ and $t_{j\neq i} = t + dt$. In that specific case, $\psi(N-i+1, H, \lambda.(t_{(i)} - t_{(i-1)}))$ is 1 except for $i = 1$ and $i = 2$, where they equal, $\psi(N, H, \lambda t)$ and $\psi(N-1, H, \lambda dt)$, respectively. This results from Lemma 2.1. Further, it is trivial to show that

$$\psi(N, H, \lambda.(t+dt)) = \psi(N, H, \lambda t)\psi(N, H, \lambda dt)$$

and one gets

$$S(t+dt,\ldots,t+dt,\underbrace{t}_{i},t+dt,\ldots,t+dt) - S(t+dt) =$$

$$\psi(N,H,\lambda t)\prod_{j\neq i} S_j(t+dt)\Big(S_i(t)\psi(N-1,H,\lambda dt) - S_i(t+dt)\psi(N,H,\lambda dt)\Big)$$

In order to evaluate the limit of the ratio of the above expression with dt as $dt \to 0$ we use the equivalent formulation of the numerator given in the lemma, and evaluate the first order expansions of the above exponentials (see Eq. 21) which directly leads to the claim.

Disclaimer

The views expressed in this work are those of the author, and do not necessarily reflect the position of ING.

References

[Chapovsky et al. (2006)] Chapovsky, A. , Rennie, 1. and Tavares P. (2006) *Stochastic Intensity Modeling for Structured Credit Exotics*, Merril Lynch research Report.

[Cont and Tankov (2004)] Cont, R. and Tankov, P. (2004) *Financial Modelling With jump Processes*, Chapman & Hall/CRC.

[Duffie and Garleanu (2001)] Duffie, D. and Garleanu, N. (2001) *Risk and Valuation of Collaterized Debt Obligation*, Financial Analyst Journal 57(1).

[Hull and White (2008)] Hull J.C. and White A. (2008) *Dynamic Models of Portfolio Credit Risk : A simplified Approach*. Journal of Derivatives, 15(4), pp. 9-28.

[Li (1999)] Li, D. (1999) *On Default Correlation : A Copula Function Approach*. Riskmetrics Working Paper 99-07.

[Mai and Scherer (2009)] Mai, J.F. and Scherer, M. (2009) *Pricing k-th to default swaps in a Lévy-time framework*, Journal of Credit Risk 5(3), pp. 55-70.

[Laurent and Gregory (2003)] Laurent, J.-P. and Gregory, J. (2003) *Basket Default Swaps, CDO's and Factor Copulas*, Journal of Risk 7(4), pp.103-122.

[Papageorgiou and Sircar (2008)] Papageorgiou, E. and Sircar, R. (2008) *Multiscale Intensity Models and Name Grouping for Valuation of Multi-Name Credit Derivatives*, Research paper, Dpt of Operations Research & Financial Engineering, Princeton University.

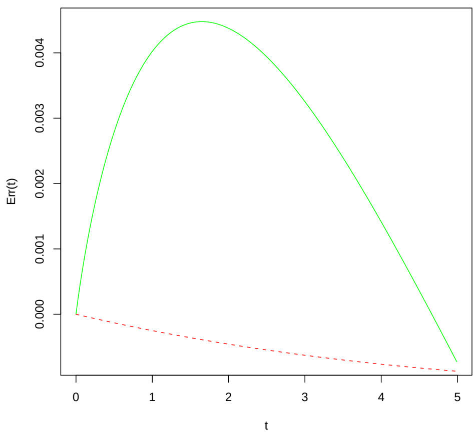
[Schoutens (2003)] Schoutens, W. (2003) *Lévy Processes in Finance*, Wiley.

[Cherubini et al. (2006)] Cherubini, U., Luciano, E. and Vecchiato, W. (2003) *Copula Methods in Finance*, Wiley.

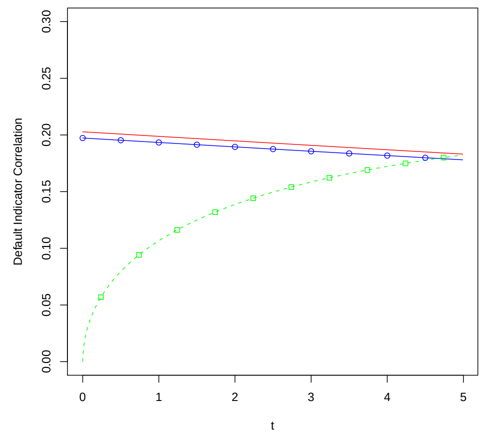
[Sibuya (1960)] Sibuya, M. (1960) *Bivariate Extreme Statistics*, Annals of the Institute of Statistical Mathematics, 11(2) pp. 195-210.

[Anderson et al. (1992)] Anderson, J.E., Louis, T.A., Holm, N.V. and Harvald, B. (1992) *Time-dependent association measures for bivariate survival distributions*, Journal of the American Statistical Association, 87(419) pp. 641-650.

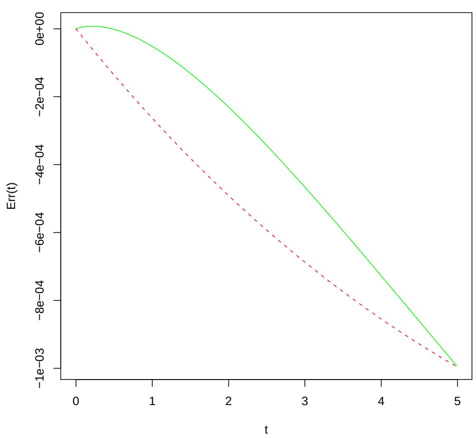
[Drouet Mari (1999)] Drouet Mari, D. (1999) *La dépendance positive entre deux variables de durées : concepts de dépendance et mesures locales de liason*, Revue de Statistique Appliquée, 47(4) pp. 5-24.



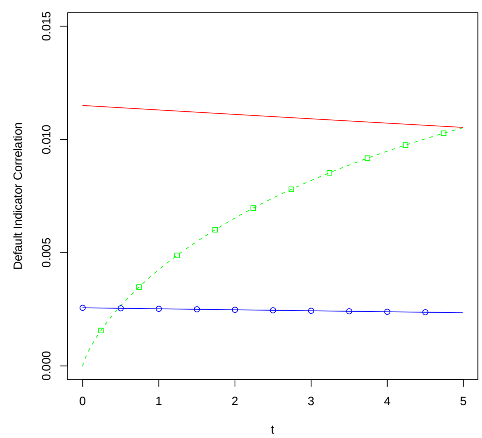
(a) Case 1 : $Err(t)$, $S_{ij}(T,T) \approx 63.8\%$



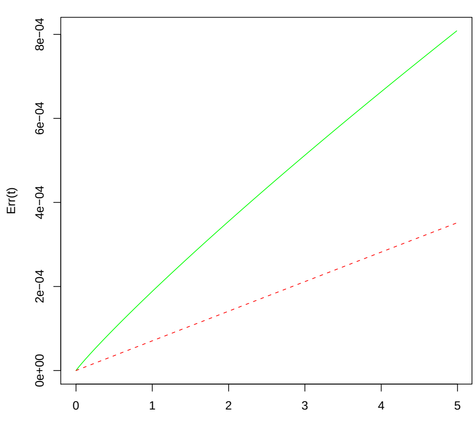
(b) Case 1 : Default correlation.



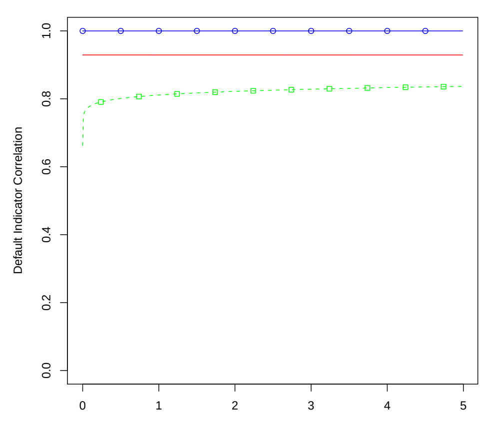
(c) Case 2 : $Err(t)$, $S_{ij}(T,T) \approx 70.6\%$



(d) Case 2 : Default correlation.

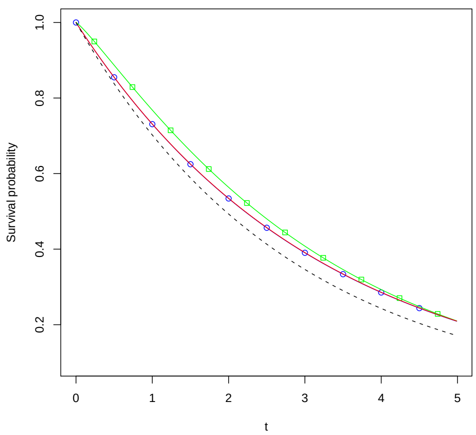


(e) Case 3 : $Err(t)$, $S_{ij}(T,T) \approx 99.5\%$

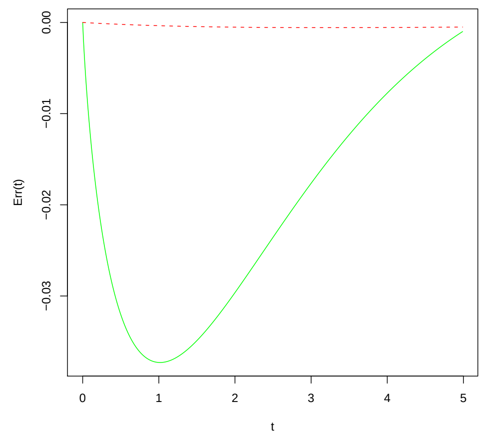


(f) Case 3 : Default correlation.

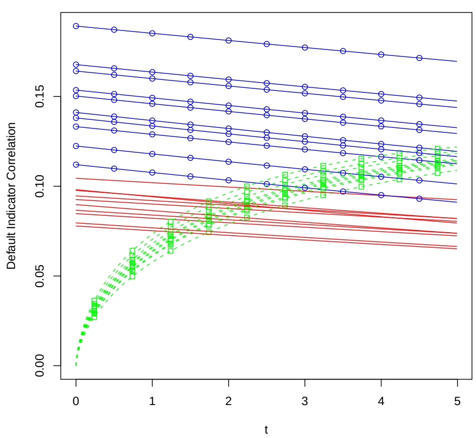
Figure 1: $Err(t)$ defined as $S^{HW}(t) - S^{GC}(t)$ (green solid) or $S^{HW}(t) - S^{MS}(t)$ (red dashed) and default correlation $\mathrm{Corr}(A_i(t), A_j(t))$ under various models under constraint that $|S^{HW}(T) - S^{GC}(T)| < \epsilon$ with $T = 5$, $\epsilon = 10E - 3$. The legend is Hull & White (blue, dots), Gaussian Copula (green, squares), Mai & Scherer with compound Poisson process$(1, \eta)$ as Lévy subordinator (red, solid).



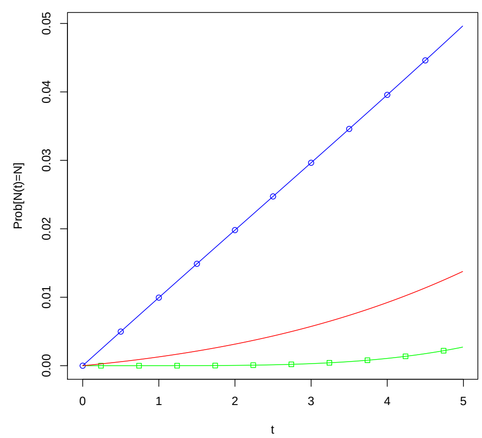
(a) Survival probability of first-to-default time



(b) $Err(t)$



(c) Pairwise default correlations



(d) Default probability of last-to-default time

Figure 2: Model impacts on a 5-names basket default swap under calibration of survival probability up to maturity $T = 5$. The legend is : Hull & White (blue, dots), Gaussian Copula (green, squares), Mai & Scherer with compound Poisson process$(1, \eta)$ as Lévy subordinator (red, solid). On panel 2(a), dashed curve is survival distribution assuming independence, and we note that the Mai & Scherer curve is almost the same as that of Hull & White.